jeudi 8 octobre 2009
11:14

Ginzburg - Landau theory

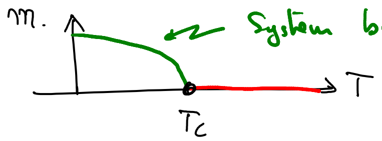
I] Basic Idea:

Singularities in the first derivatives of $F \to$ first order transition

2^{nd}. $F \to$ second order "

System has a symmetry $H[m \to -m]$

Magnet.



System breaks (spontaneously) a symmetry.

Phenomenology: $\to$ Systems with spontaneous symmetry breaking

$\to$ transition is of second order.

Order parameter.

zero in $T > T_c$

non zero in $T < T_c$

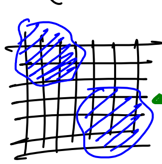
} characterizes the breaking of symmetry

phenomenology: m is continuous at the transition (2^{nd} order)

$F[T]$ free energy of the system.

$\mathcal{F}[T, m(x)]$: Landau functional.

$$Z = \sum_{\sigma} e^{-\beta H[\sigma]} = \sum_{\sigma} \sum_{m(x)} \delta(\text{"}\sigma\text{"} - m(x))\, e^{-\beta H[\sigma]}$$

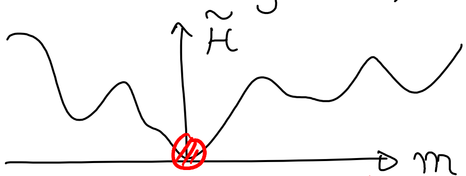


m(x)

$$= \sum_{m(x)} e^{-\beta \tilde{\mathcal{H}}(m(x))}$$

$$Z = \int \mathcal{D}m(x)\; e^{-\beta \tilde{\mathcal{H}}(m(x))}$$

$\tilde{\mathcal{H}}$



↑ largest "Boltzmann" weight

assume that 1 configuration dominates

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minimize $\tilde{\mathcal{H}}(m(x)) \longrightarrow m^*(x)$

$$Z = \sum_{m(x)} e^{-\beta \tilde{\mathcal{H}}(m)} \approx e^{-\beta \tilde{\mathcal{H}}[m^*(x)]}$$

$$F = -\frac{1}{\beta} \log Z = \tilde{\mathcal{H}}[m^*(x)] \Leftarrow \tilde{\mathcal{H}} \Leftrightarrow \mathcal{F}$$

(Landau Functional)

$\mathcal{F}[m(x)]$ minimize it $\rightarrow$ $m^*(x)$, $F[T] = \mathcal{F}[m^*(x)]$

Guess. $\mathcal{F}[m(x)]$

$T > T_c$ $\quad F = F_0[T] \Leftarrow$ normal system

$T \approx T_c$. $\quad m(x)$ is small

$$\mathcal{F}[m(x)] = F_0[T] + \cancel{m(x)} + m(x)\, m(x')$$

assumes m is nearly uniform $\qquad A \; m(x)^2$

$$F_0[T] + A\int dx \; m(x)^2 + B\int dx \; m^4(x).$$

$m(x) \rightarrow m$. $\qquad \Omega\left[A(T) m^2 + B(T) m^4 \right]$

One needs $B > 0$

$A > 0 \quad \rightarrow \quad m^* = 0$

at $T = T_c \quad A = 0 \qquad T < T_c \quad A < 0$

- The simplest (most generic case) $\quad A(T) = \alpha (T - T_c)$

$B(T) \simeq B(T_c) \quad (\neq 0 \;\; > 0)$

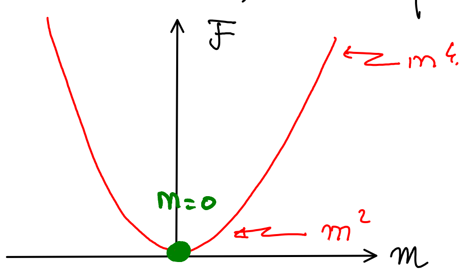
$$\mathcal{F}[m(x), T] = F_0[T] + \alpha (T - T_c) \int dx \; m(x)^2 + \beta \int dx \; m^4(x) + \frac{\gamma}{2} \int (\nabla m(x))^2$$

$\alpha, \beta, \gamma > 0$

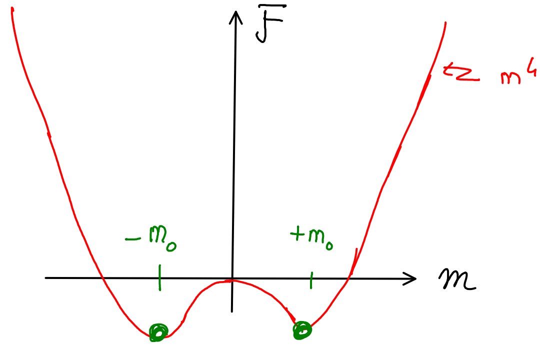
Uniform m:

$$\mathcal{F}[m,T] = F_0[T] + \alpha(T-T_c)\, m^2\, \Omega + \beta\, \Omega\, m^4.$$

$$\alpha(T-T_c)\, m^2 + \beta m^4.$$



$T > T_c$



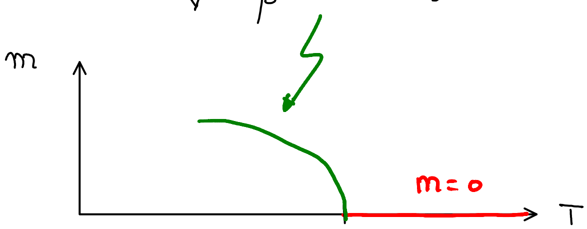
$T > T_c$

$$f(m) = \alpha(T-T_c)\, m^2 + \beta m^4$$

$$\frac{\partial f(m)}{\partial m} = 2\alpha(T-T_c)\, m + 4\beta m^3 = 0$$

$$\begin{cases} m = 0 \\ 2\alpha(T-T_c) + 4\beta m^2 = 0 \end{cases} \Rightarrow m^2 = \frac{-\alpha}{2\beta}(T-T_c)$$

$$m_0 = \pm\sqrt{\frac{-\alpha}{2\beta}(T-T_c)}$$



$$m \propto \sqrt{T_c - T}$$

$$m \propto |T-T_c|^{1/2} \leftarrow \text{Critical exponent.}$$

GL predicts $m \propto |T-T_c|^{1/2}$.

- Free energy

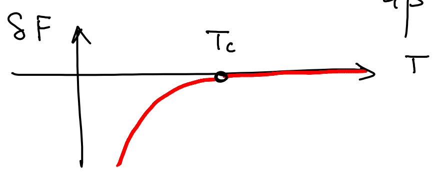
$T > T_c \qquad m^* = 0 \qquad \Rightarrow \qquad F = \mathcal{F}[m^*=0, T] = F_0[T]$

$T < T_c \qquad m^* = \sqrt{\frac{-\alpha}{2\beta}(T-T_c)} \qquad \Rightarrow \qquad F = \mathcal{F}[m^*, T] = F_0[T] + \Omega[\alpha(T-T_c)m^2 + \beta m^4]$

$$= F_0[T] + \Omega\, m^{*2}\left[\alpha(T-T_c) + \beta\left(\frac{-\alpha}{2\beta}\right)(T-T_c)\right]$$

$$= F_0[T] + \Omega\left[\frac{-\alpha}{2\beta}(T-T_c)\right]\left[\frac{\alpha}{2}(T-T_c)\right]$$

$$= F_0[T] - \Omega \frac{\alpha^2}{4\beta}(T-T_c)^2 \qquad \underline{T<T_c}$$



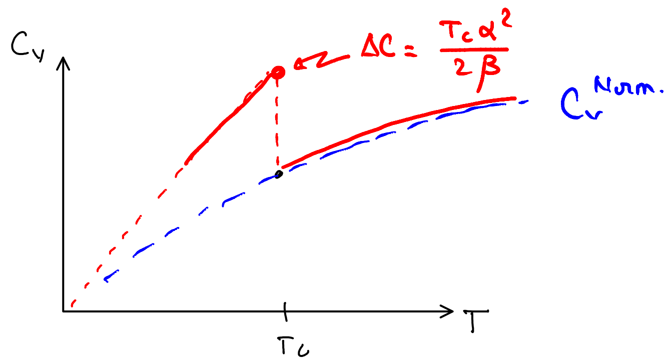
$$S = -\frac{\partial F}{\partial T} \qquad T\,ds = dQ \qquad C_V = \frac{dQ}{dT} = T\frac{ds}{dT}$$

$$C_V = -T\frac{\partial^2 F}{\partial T^2}.$$

$$T>T_c \Rightarrow C_V = C_V^{\text{Norm.}} \left(= -T\frac{\partial^2 F_0}{\partial T}\right)$$

$$T<T_c = C_V = C_V^{\text{Norm.}} + \left[-T\frac{\partial^2}{\partial T} - \frac{\Omega\alpha^2}{4\beta}(T-T_c)^2\right]$$

$$C_V = C_V^{\text{Norm}} + T\frac{\Omega\alpha^2}{2\beta}$$



$$C_V \sim |T-T_c|^{-\alpha}.$$

"$\alpha = 0$"

Effect of a magnetic field

$$\mathcal{F}[m,T] = F_0[T] + \alpha(T-T_c)\int m^2 + \beta\int m^4 + \frac{\gamma}{2}\int(\nabla m)^2 - h\int d^dx\, m(x)$$

uniform m

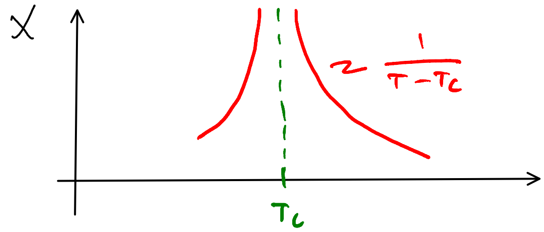
$$\mathcal{F}[m,T] = F_0[T] + \Omega\left[\alpha(T-T_c)m^2 + \beta m^4 - hm\right]$$

$$\frac{\partial\mathcal{F}}{\partial m} = 0 \Rightarrow 2\alpha(T-T_c)m + 4\beta m^3 - h = 0$$

$$\chi = \left.\frac{\partial m}{\partial h}\right|_{h\to 0}. \qquad T>T_c \qquad 2\alpha(T-T_c)m - h \simeq 0$$

$$m = \frac{h}{2\alpha(T-T_c)}$$

$$\chi = \frac{1}{2\alpha(T-T_c)} \propto \frac{1}{(T-T_c)^\gamma} \qquad \boxed{\gamma = 1}$$



Continuous symmetry:

$\vec{m}$ $\quad H[\vec{m}]$ rotations of $\vec{m}$

$$\mathcal{F} = F_0 \; \cancel{\vec{m}} + \alpha(T-T_c)\,\vec{m}\cdot\vec{m} + \beta(\vec{m}\cdot\vec{m})^2$$

Superconductivity

$\Psi(x)$ "superconducting" wavefunction complex

$U(1)$ symmetry $\quad e^{i\theta}\psi \rightarrow$ physics is invariant

$$\mathcal{F} = F_0 + \alpha(T-T_c)\int d^dx\; \psi^*(x)\psi(x) + \beta\int d^dx\,(\psi^*\psi(x))^2 + \frac{\gamma}{2}\int \nabla\psi^*\,\nabla\psi\; d^dx$$

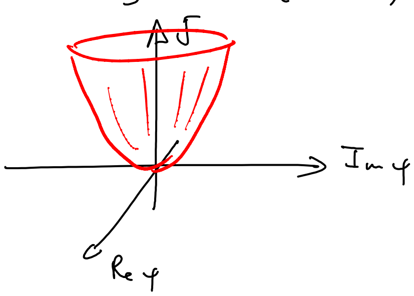
[Tinkham on Superconductivity]

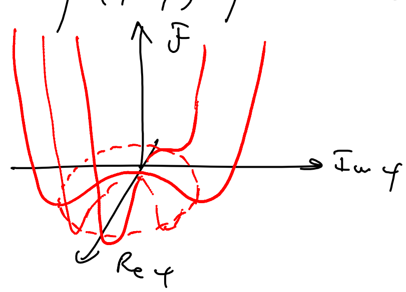
$$f(\psi,\psi^*) = \alpha(T-T_c)\,\psi^*\psi + \beta(\psi^*\psi)^2 .$$

$$\frac{\partial f}{\partial \mathrm{Re}\,\psi} \quad \frac{\partial f}{\partial \mathrm{Im}\,\psi} \longleftrightarrow \frac{\partial f}{\partial \psi} \quad \frac{\partial f}{\partial \psi^*}$$

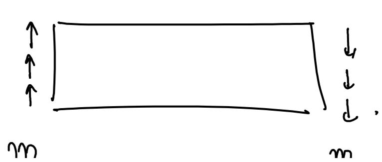
$$\frac{\partial f}{\partial \psi} = 0 \rightarrow \alpha(T-T_c)\,\psi^* + 2\beta(\psi^*\psi)\,\psi^* = 0$$

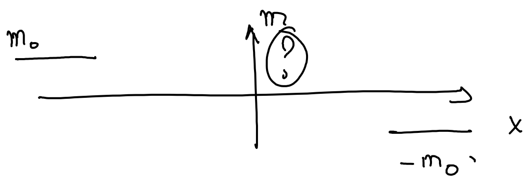
$$\frac{\partial f}{\partial \psi^*} = 0 \rightarrow \alpha(T-T_c)\,\psi + 2\beta(\psi^*\psi)\,\psi = 0$$

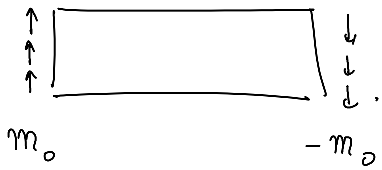


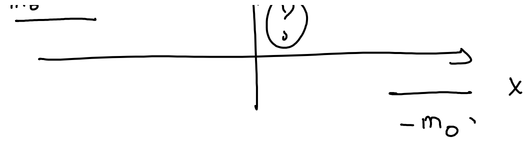


Variations of the order parameter







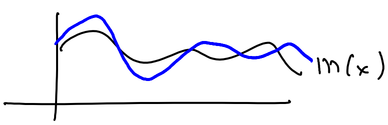


$$F[m(x),T] = F_0[T] = \alpha(T-T_c)\int d^dx\; m(x)^2 + \beta\int d^dx\; m(x)^4$$
$$+\frac{\gamma}{2}\int d^dx\,(\nabla m(x))^2$$

$$\frac{\partial F[m(x)]}{\partial m(x_1)} \qquad F[m_1, m_2, m_3, m_4, m_5 \cdots\cdots m_n]$$

$$\frac{\partial F[m_1 \cdots\cdots m_N]}{\partial m_j} \qquad \frac{\partial m_i}{\partial m_j} = \delta_{i,j}$$

$$\frac{\partial F[m(x)]}{\partial m(x_1)} \qquad \frac{\partial m(x_i)}{\partial m(x_j)} = \delta(x_i - x_j)$$



$$m(x) \to m(x) + \delta m(x) \qquad \delta m \text{ small}$$

$$f(x+\varepsilon) - f(x) = \varepsilon \left.\frac{\partial f}{\partial x}\right|_x$$

$$f(x_1+\varepsilon_1, x_2+\varepsilon_2, \cdots) - f(x_1, \cdots x_n) = \sum_i \varepsilon_i \left.\frac{\partial f}{\partial x_i}\right|_{\varepsilon=0}$$

$$F[m(x)+\delta m(x)] - F[m(x)] = \int d^dx\; \delta m(x)\,\frac{\partial F[m(x)]}{\partial m(x)}$$

$$F[m(x)+\delta m(x)] - F[m(x)] =$$
$$\alpha(T-T_c)\int d^dx\,\left[(m(x)+\delta m(x))^2 - (m(x))^2\right]$$
$$+\beta \int d^dx\left[(\qquad)^4 - (\quad)^4\right]$$
$$+\frac{\gamma}{2}\int d^dx\; \nabla(m(x)+\delta m(x))\,\nabla(m(x)+\delta m(x)) - (\nabla m)^2.$$

$$= \alpha(T-T_c)\int d^dx\; \delta m(x)\; 2m(x)$$
$$+\beta\int d^dx\; \delta m(x)\; 4\,m(x)^3$$

$$+ \frac{\gamma}{2} \int d^d x \; 2 \, \nabla \delta m(x) \, \nabla m(x)$$

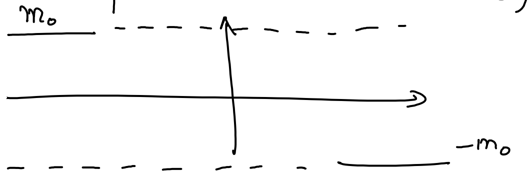
$$= \int d^d x \; \delta m(x) \left[\alpha (T - T_c) \, 2 m(x) + 4 \beta \, m(x)^3 \right]$$

$$- \frac{\gamma}{2} \int d^d x \; 2 \, \delta m(x) \, \nabla^2 m(x) \qquad \text{at } \pm\infty \quad \nabla m(x) \to 0$$

$$\frac{\partial \mathcal{F}[m(x)]}{\partial m(x)} = 2 \alpha (T - T_c) \, m(x) + 4 \beta \, m^3(x) - \gamma \nabla^2 m(x)$$

$$2 \alpha (T - T_c) \, m(x) + 4 \beta \, m^3(x) - \gamma \nabla^2 m(x) = 0$$ | optimal configuration.

m uniform. $\quad 2\alpha (T - T_c) \, m + 4 \beta \, m^3 \quad \to \quad m^2 = \frac{-\alpha}{2\beta} (T - T_c)$



$$m(x) = m_0 \, f(x)$$

$$2 \alpha (T - T_c) \, m_0 \, f(x) + \underbrace{4 \beta \, m_0^3}_{-2\alpha (T - T_c) m_0 \, .} f^3(x) - \gamma \, m_0 \nabla^2 f(x) = 0$$

$$2 \alpha (T - T_c) \cancel{m_0} \left[f(x) - f^3(x) \right] - \gamma \cancel{m_0} \nabla^2 f(x) = 0$$

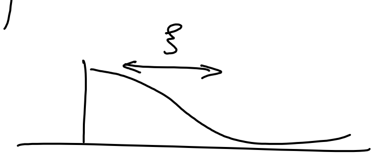
$$f(x) - f^3(x) = \frac{\gamma}{2\alpha (T - T_c)} \nabla^2 f(x) = \frac{\gamma}{2\alpha (T - T_c)} \frac{\partial^2 f}{\partial x^2} \, .$$

$f = \pm 1$ are solutions.

$$\frac{\gamma}{2\alpha (T - T_c)} \equiv \xi^2$$

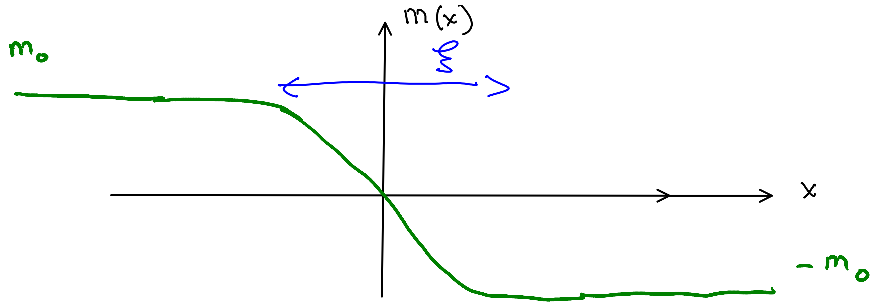
|| Characteristic length of variation of the order parameter.

$$\xi \sim \frac{1}{|T - T_c|^{\nu}} \qquad \nu = 1 .$$

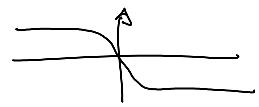
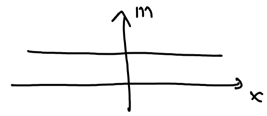


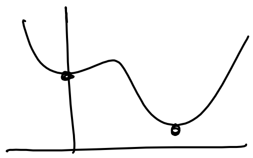
$$f(y) - f^3(y) = \frac{\partial^2 f(y)}{\partial y^2} \qquad y = x / \xi$$

$$\frac{d^2 r}{d t^2} = F(r) \qquad\qquad f(x) = \pm \tanh\left(\frac{x}{\sqrt{2}\,\xi} \right)$$



domain wall.

Q: is there an energy cost of  



$$F[m(x),T] = F_0[T] = \alpha(T-T_c)\int d^dx\; m(x)^2 + \beta\int d^dx\; m(x)^4$$
$$+ \frac{\gamma}{2}\int d^dx\; (\nabla m(x))^2$$

$$F[m_1(x),T] - F[m_0,T] =$$
$$\alpha(T-T_c)\int d^dx\; m_0^2\,[f^2-1]$$
$$+ \beta \int d^dx\; m_0^4\,[f^4-1]$$
$$+ \frac{\gamma}{2}\int d^dx\; m_0^2(\nabla f)^2$$

III] beyond G.L.:

$$Z = \int \mathcal{D}m(x)\; e^{-\beta F[m(x),T]} \longrightarrow e^{-\beta F[m^*(x),T]}$$

$$F[m(x),T] = \alpha(T-T_c)\int d^dx\; m(x)^2 + \beta\int d^dx\; m^4(x)$$
$$+ \frac{\gamma}{2}\int d^dx\; (\nabla m)^2 .$$

$$F = -\frac{1}{\beta}\log Z$$

$$Z = \int \mathcal{D}m(x)\; e^{-\left[\frac{\gamma}{2}\int d^dx\,(\nabla m)^2 + \frac{1}{2}\alpha(T-T_c)\int d^dx\, m^2(x) + b\int d^dx\, m^4(x)\right]}$$

Can one expand?

$$G^{-1}(q) = \left[\gamma q^2 + \alpha(T-T_c)\right]$$

$T > T_c$ $$e^{-\frac{1}{2\Omega}\sum_q \left[\gamma q^2 + \alpha(T-T_c)\right] m_q^* m_q}$$

$$\int \mathcal{D}m \; e^{-\frac{1}{2}\sum_{ij} m_i M^{-1}_{ij} m_j}$$ all the eigenvalues positive

$$\int_{-\infty}^{+\infty} dx \; e^{-ax^2} \qquad \int dx \; e^{+ax^2} \quad !$$

$T \gg T_c$. $\int dx \; e^{-ax^2}$ a is large.

$\langle x^2 \rangle \sim \frac{1}{a}$ is small. $\to$ perturbation theory is very well. Convergent.

$T - T_c \gg T_c$. $\Rightarrow$ perturbation theory works very well.

$$\langle \Theta \rangle = \langle \Theta \rangle_{b=0} + b \langle \Theta \, m^4 \rangle + \cdots$$

$$\int \mathcal{D}\phi \, e^{-(T-T_c)\phi^2} \quad \phi^2 \sim \frac{1}{(T-T_c)^{3/2}} \qquad \langle m^4 \rangle \sim \frac{1}{(T-T_c)^3}$$

$$\tilde{\phi} = \sqrt{T-T_c}\,\phi$$

$$\frac{\langle O \rangle_b}{\langle \Theta \rangle_{b=0}} = 1 + b\frac{1}{(T-T_c)^3} \qquad (T-T_c) \gg T_c$$

Perturbation theory.

$T - T_c$ is fixed and $b \to 0$ perturbation is well defined

Physics in CM.

b is fixed by the system

we want the physics when $T \to T_c$.

Saddle point approximation.

$$\mathcal{F}[\phi(x)] = \int dx \; \frac{1}{2} a(T) \phi^2(x) + b\,\phi^4(x) + \frac{c}{2}(\nabla\phi)^2$$

$$- \int h(x)\, \phi(x) \, dx$$

$$Z = \int \mathcal{D}\phi \; e^{-N \mathcal{F}[\phi]} \qquad N \text{ large.}$$

$$\frac{\partial F}{\partial \phi} = 0 \qquad a(T)\,\phi + 4b\,\phi^3(x) - h = 0$$

$$T > T_c \qquad \phi^* \approx \frac{h}{a(T)} \qquad a(T) = 0 \quad T = T_c.$$

$$\chi = \frac{1}{T - T_c}$$

$$\mathcal{F}[\phi^* + \delta\phi(x)] = \mathcal{F}[\phi^*] + \int dx \left[\frac{1}{2} a(T)\, \delta\phi^2(x) + \frac{c}{2} (\nabla \delta\phi)^2 + 6b\, \phi^{*2}\, \delta\phi^2(x) \right]$$

$$\mathcal{F}[\phi^* + \delta\phi(x)] = \mathcal{F}[\phi^*] + \frac{1}{2} \int dx \left[a(T)\, \delta\phi^2 + c (\nabla\delta\phi)^2 + 12 b\, \phi^{*2} \delta\phi^2 \right]$$

$$Z = \int \mathcal{D}\phi \; e^{-N\mathcal{F}[\phi^*] - N\delta\mathcal{F}[\delta\phi]} = \underbrace{e^{-N\mathcal{F}[\phi^*]}}_{Z_{GL}} \underbrace{\int \mathcal{D}\delta\phi \; e^{-N\delta\mathcal{F}[\delta\phi]}}_{\delta Z}$$

$$F = -\frac{1}{\beta} \log Z = F_{GL} + \delta F$$

$$\chi = \frac{\partial m}{\partial h} = \frac{\partial^2 F}{\partial h} = \chi_{GL} + \delta\chi \qquad \frac{\partial^2 \delta F}{\partial h^2}$$

$$\delta Z = \int \mathcal{D}\phi \; e^{-\frac{N}{2} \int [\qquad]}$$

$$= \text{Cst} \int \mathcal{D}\tilde{\phi}_q \mathcal{D}\tilde{\phi}_q^* \; e^{-\frac{1}{2\Omega} \sum_q \left[a(T) + 12 b\, \phi^{*2} + c q^2 \right] \phi_q^* \phi_q}$$

$$\delta F = -\frac{1}{\beta} \log \delta Z = -\frac{1}{\beta} \sum_{q>0} \log \left[\frac{\pi}{a(T) + 12 b\, \phi^{*2} + c q^2} \right]$$

$$\delta\chi = \frac{\partial^2 \delta F}{\partial h^2} \qquad \phi^* = \frac{h}{a(T)} \qquad h = a(T)\, \phi^*$$

$$= \frac{1}{a(T)^2} \frac{\partial^2 \delta F}{\partial \phi^{*2}}$$

$$\chi = \underbrace{\frac{1}{a(T)}}_{\chi_{GL}} - \frac{1}{a(T)^2}\sum_{q>0} \frac{12b}{a(T) + 12b\,\phi^{*2} + cq^2}\Bigg|_{h=0}$$

$$\chi = \frac{1}{a(T)} - \frac{1}{a(T)^2}\sum_{q>0}\frac{12b}{a(T) + cq^2}$$

$$\chi = \frac{1}{a(T)}\left[1 - \frac{1}{a(T)}\sum_{q>0}\frac{12b}{a(T)+cq^2}\right]$$

$$\chi^{-1} = a(T)\left[1 - \frac{1}{a(T)}\sum_{q>0}\frac{12b}{(\qquad)}\right]^{-1}$$

$$\chi^{-1} = a(T) + \text{Cst}\sum_{q>0}\frac{b}{a(T)+cq^2}. \qquad a(T) = \alpha(T - T_c^0)$$

T_c^0 : critical temperature for GL

Correction to T_c due to the fluctuations around the saddle point (GL)

$\chi^{-1} = 0$

$$0 = \alpha(T - T_c^0) + \text{cst}\sum_{q>0}\frac{b}{\underbrace{\alpha(T-T_c^0)}_{\text{at this order } = 0} + cq^2}.$$

$$0 = \alpha(T - T_c^0) + \text{Cst}\cdot\sum_{q>0}\frac{b}{cq^2}.$$

$$0 = T - T_c^0 + \frac{\text{Cst}}{\alpha}\sum_q \frac{b}{cq^2}.$$

$$T_c = T_c^0 - \frac{\text{Cst}}{\alpha}\sum_{q>0}\frac{b}{cq^2}. = T_c^0 - \frac{\text{Cst}}{\alpha}\int\frac{d^dq}{(2\pi)^d}\frac{b}{cq^2}.$$

fluctuations decrease the T_c $\qquad T_c < T_c^0$

$d \leq 2$ Correction diverges $\Rightarrow$ no ordered state at finite temperature.

$(\nabla\phi)^2$ variation $\rightarrow$ $q^2\phi_q^*\phi_q$

[Goldstone modes]

$$\chi^{-1} = \overset{\alpha(T-T_c^0)}{a(T)} + \mathrm{Cst} \sum_{q>0} \frac{b}{\underbrace{a(T)}_{0 \text{ at this order}} + c q^2}$$

$T \simeq T_c$

$$0 = \alpha(T_c - T_c^0) + \mathrm{Cst} \sum_{q>0} \frac{b}{c q^2}$$

$$\chi^{-1} = \alpha(T - T_c) + \mathrm{Cst} \sum_{q>0} \left[\frac{b}{a(T) + c q^2} - \frac{b}{c q^2} \right]$$

$$= \alpha(T - T_c) + \mathrm{Cst} \sum_{q>0} \frac{-a(T)\, b}{[a(T) + c q^2][c q^2]}$$

$$\chi^{-1} = \alpha(T - T_c) - \mathrm{Cst}\; a(T) \sum_{q>0} \frac{b}{[c q^2]^2}$$

$$= \alpha(T - T_c) - \mathrm{Cst}\; \alpha(T - T_c) \sum_{q>0} \frac{b}{[c q^2]^2}$$

$$= \alpha(T - T_c) \left[1 - \mathrm{Cst} \sum_{q>0} \frac{b}{[c q^2]^2} \right]$$

$\underline{d > 4}$ the only effect of the fluctuations

→ lower slightly T_c

→ change the amplitude of χ

don't change the exponent

$\underline{d \leq 4}$ fluctuations give a divergent correction!

$$1 - \underbrace{b \sum_{q \sim 1/L}^{\pi/a} \frac{1}{q^4}}$$ $\qquad q^{d-4} \quad \left(\frac{1}{q}\right)^{4-d}$

$$1 - b \left(\frac{L}{a}\right)^{4-d}$$

L : size of the system.
length of the observable

$$L^{\nu} = e^{\nu \operatorname{Log} L} = 1 + \nu \operatorname{Log} L + \frac{1}{2} \nu^2 \operatorname{Log}^2 L + \dots$$