jeudi 15 octobre 2009
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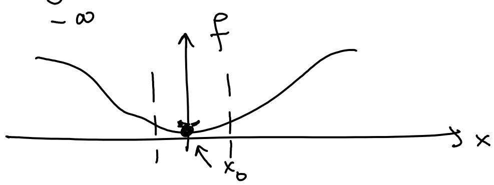
Methods of Computation of functional Integrals

I] Saddle point approximation

$$I = \int_{-\infty}^{+\infty} dx \quad e^{-N f(x)} \qquad\qquad N \to \infty$$

dominated by $e^{-N f(x_0)}$



$$f(x) \approx f(x_0) \qquad\qquad I \approx e^{-N f(x_0)}$$

$$e^{-N [\underbrace{f(x_1) - f(x_0)}_{>0}]} \longrightarrow 0 \qquad\qquad \left.\frac{\partial f}{\partial x}\right|_{x_0} = 0$$

$$f(x) = f(x_0) + \frac{1}{2}(x-x_0)^2 \left.\frac{\partial^2 f}{\partial x}\right|_{x_0}$$

$$I = \int_{-\infty}^{+\infty} dx \quad e^{-N\left[f(x_0) + \frac{1}{2}(x-x_0)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0} + \dots\right]}$$

$$= e^{-N f(x_0)} \left[\int_{x_0-a}^{x_0+a} dx \quad e^{-\frac{N}{2}(x-x_0)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{x_0}}\right]$$

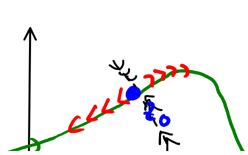
$$= e^{-N f(x_0)} \int_{-a}^{a} dx' \quad e^{-\frac{N}{2}\left.\left(\frac{\partial^2 f}{\partial x^2}\right)\right|_{x_0} (x')^2}.$$

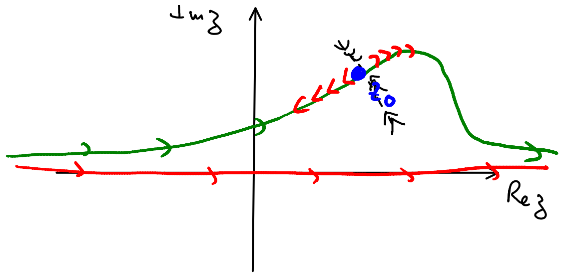
$$= e^{-N f(x_0)} \int_{-a\sqrt{N}}^{+a\sqrt{N}} dy \quad e^{-\frac{1}{2}\left.\left(\frac{\partial^2 f}{\partial x^2}\right)\right|_{x_0} y^2}.$$

$$\longrightarrow e^{-N f(x_0)} \int_{-\infty}^{+\infty} dy \quad e^{-\frac{1}{2}\left.\left(\frac{\partial^2 f}{\partial x^2}\right)\right|_{x_0} y^2} = e^{-N f(x_0)} \sqrt{\frac{\pi}{\left.\left(\frac{\partial^2 f}{\partial x^2}\right)\right|_{x_0}}}$$

$$\int_{-\infty}^{+\infty} dx \quad e^{-N f(x)} \qquad\qquad f(x) \text{ no minimum on } \mathbb{R}$$

$f(z)$ has an extremum in the complex plane



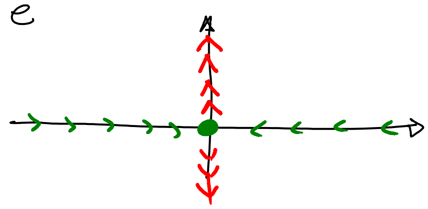


$$I = \int_{-\infty}^{+\infty} e^{-Nf(x)}\, dx$$

$$\oint dz\, f(z) = 2i\pi \sum_{\text{poles}} \text{residues } f.$$

$$I = e^{-Nf(z_0)}$$

$$e^{Nx^2}$$

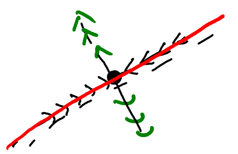


$$\frac{\partial f}{\partial x} = 0 = 2x \qquad x = 0$$

$$e^{Nz^2} \qquad z = iy \qquad e^{-Ny^2}$$

$$I = e^{-Nf(z_0)} \int_{-\infty}^{+\infty} dy\; e^{-N\left(\frac{\partial^2 f}{\partial y^2}\right) y^2}$$

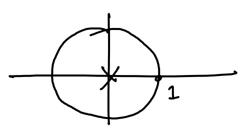
Find an extremum z_0



Example of Saddle point:

$$\frac{1}{(p-1)!} = \frac{1}{2i\pi} \oint dz\; e^{z} \frac{1}{z^p} \qquad e^{z} = \sum_{n=0}^{+\infty} \frac{1}{n!} z^n$$

$$A = \frac{1}{2i\pi} \oint dz\, \frac{z^n}{z^p}.$$

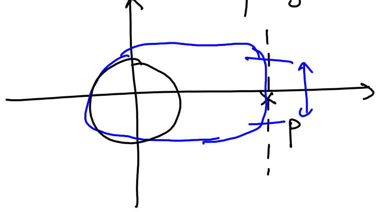


$$z = e^{i\theta} \qquad dz = e^{i\theta}\, i\, d\theta = z\, i\, d\theta$$

$$A = \frac{1}{2\pi} \int_0^{2\pi} d\theta\, \frac{z^{n+1}}{z^p} = \frac{1}{2\pi} \int_0^{2\pi} d\theta\; e^{i[[n+1]-p]\theta} = \delta_{n+1,p}.$$

$$\frac{1}{(p-1)!} = \frac{1}{2i\pi} \oint dz\; e^{z} \frac{1}{z^p} = \frac{1}{2i\pi} \oint dz\; e^{z - p\log z}$$

$$F(z) = z - p\log z. \qquad \frac{\partial F(z)}{\partial z} = 0 = 1 - \frac{p}{z}. \Rightarrow z = p$$



$$F(z) = (z_0 + \delta z) - p\log[z_0 + \delta z] = p + \delta z - p\log p - p\log\left[1 + \frac{\delta z}{p}\right]$$

$$F(z) = (z_0 + \delta z) - p \operatorname{Log}[z_0 + \delta z] = p + \delta z - p \operatorname{Log} p - p \operatorname{Log}\left[1 + \frac{\delta z}{p}\right]$$

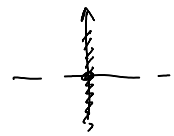
$$= p - p \operatorname{Log} p + \delta z - p\left[\frac{\delta z}{p} - \frac{1}{2}\left(\frac{\delta z}{p}\right)^2\right]$$

$$= p - p \operatorname{Log} p + \frac{1}{2} p \left(\frac{\delta z}{p}\right)^2 .$$

$$I = \frac{1}{2i\pi} \oint dz \, e^{z} \frac{1}{z^p} = \frac{1}{2i\pi} e^{p - p\operatorname{Log} p} \int d\tilde{z} \; e^{\frac{1}{2} p \left(\frac{\delta \tilde{z}}{p}\right)^2}$$

$$\int_{-\infty}^{+\infty} i \, dy \; e^{-\frac{1}{2} p \left(\frac{y}{p}\right)^2}$$

Minimum $\tilde{z} = iy$



$$I = \frac{1}{2\pi} e^{p - p\operatorname{Log} p} \underbrace{\int_{-\infty}^{+\infty} dy \; e^{-\frac{1}{2} p y^2}}_{\sqrt{2\pi p}} . \qquad \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} dy \, e^{-y^2 a} = \sqrt{\frac{\pi}{a}} .$$

$$I = e^{p - p\operatorname{Log} p} \sqrt{\frac{p}{2\pi}} = e^{p - p \operatorname{Log} p} \frac{1}{\sqrt{2\pi p}} \, p = \frac{1}{(p-1)!}$$

$$(p-1)! = \frac{1}{p} \sqrt{2\pi p} \; e^{p \operatorname{Log} p - p} . = p^p \, e^{-p} \cdot \sqrt{2\pi p} \; \frac{1}{p}$$

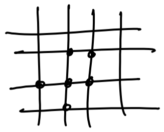
$$p! = \left(\frac{p}{e}\right)^p \sqrt{2\pi p}$$

$$[\ GL \rightarrow \quad I = \int \mathcal{D} m(x) \; e^{-N \int m^2 + m^4 + (\nabla m)^2} \qquad N \rightarrow \infty$$

$$\frac{\partial \mathcal{F}}{\partial m} = 0 \qquad I \simeq e^{-N \mathcal{F}[m^*, T]}$$

<u>Example 2:</u> Spherical model.

$$H = -J \sum_{\langle ij \rangle} S_i S_j . \qquad S_i \in [-\infty, +\infty]$$



$$Z = \prod_i \int \mathcal{D} S_i \; e^{-\beta H[S]}$$

$$\sum_{i=1}^{N} S_i^2 = N \qquad \text{Constraint}$$

$$Z = \prod \int \ldots \; e^{-\beta H[S]} \; \delta\left(\sum S^2 \ldots\right)$$

$$Z = \prod_i \int \mathcal{D}S_i \quad e^{-\beta H[S]} \quad \delta\left(\sum_i S_i^2 - N\right)$$

$$\delta(x) = \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} \quad e^{i\lambda(x)}$$

$$Z = \prod_i \int \mathcal{D}S_i \int \frac{d\lambda}{2\pi} e^{\beta J \sum_{\langle ij \rangle} S_i S_j \; + \; i\lambda\left(\sum_i S_i^2 - N\right) + \beta h \sum_i S_i}$$

$$= \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} \quad \prod_i \int_{-\infty} \mathcal{D}S_i \quad e^{-i\lambda N} \quad e^{\frac{1}{2}\sum_{ab} S_a M_{ab} S_b \; + \sum_a \beta h_a S_a}$$

$$M_{ab} = \left[\beta J_{ab} + i\lambda \delta_{ab}\right]$$

$$\prod_a \int \mathcal{D}S_a . \quad e^{\frac{1}{2}\sum_{ab} S_a M_{ab} S_b \; + \sum_a \beta h_a S_a .} \qquad \sqrt{\det M} \; e^{-\frac{1}{2}\beta h M^{-1} \beta h .}$$

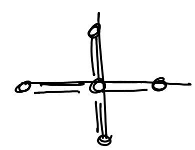
$$Z = \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} \quad e^{-i\lambda N} \quad e^{-\frac{1}{2}\operatorname{Log}\operatorname{Det} M} \quad e^{\frac{1}{2}\beta^2 \sum_{ab} h_a (M^{-1})_{ab} h_b}$$

$$= \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} \quad e^{N\left[-i\lambda - \frac{1}{2N}\operatorname{Log}\operatorname{Det} M\right]} \quad e^{\frac{1}{2}\beta^2 \sum_{ab} h_a (M^{-1})_{ab} h_b}$$

$$\chi = -\frac{\partial^2 F}{\partial h^2}\Bigg|_{h=0} = -\beta \; \frac{\int \frac{d\lambda}{2\pi} \; \sum_{ab} (M^{-1})_{ab} \; e^{N\left[-i\lambda - \frac{1}{2N}\operatorname{Log}\operatorname{Det} M\right]}}{\int \frac{d\lambda}{2\pi} \; e^{N\left[-i\lambda - \frac{1}{2N}\operatorname{Log}\operatorname{Det} M\right]}}$$

$$F = -\frac{1}{\beta}\operatorname{Log} Z$$

$$M_{ab} = \beta J_{ab} + 2i\lambda \delta_{ab}$$



$$M_{q_1 q_2} = \sum_{ab} e^{i(q_1 r_a - i q_2 r_b)} \quad M_{ab}$$

$r_a = r_b + r$

$$= \delta_{q_1 q_2} \sum_r e^{i q_1 r} \quad M_{r,0}$$

$$\operatorname{Log}\operatorname{Det} M = \operatorname{Tr}\operatorname{Log} M$$

$$\frac{1}{N}\operatorname{Tr}\left[\operatorname{Log} M\right] = \frac{1}{N}\sum_q \operatorname{Log} M(q) \Rightarrow \frac{1}{(2\pi)^d}\int d^d q \; \operatorname{Log}\left[M(q)\right]$$

1st Brillouin zone

$-\pi/a < q < \pi/a$.

1st Brillouin zone
$-\pi/a < q_{x,y,z} < \pi/a$.

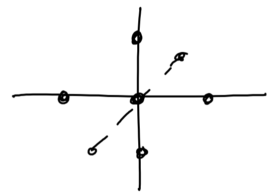
$$e^{N\left[-i\lambda - \frac{1}{2N}\operatorname{Log}\operatorname{Det} M\right]} = e^{N\left[-i\lambda - \frac{1}{2}\frac{1}{(2\pi)^d}\int_{BZ} d^dq \, \operatorname{Log}[M(q)]\right]}$$

$N \to \infty$ Saddle point becomes exact

$$\frac{\partial}{\partial\lambda}\left[-i\lambda - \frac{1}{2}\frac{1}{(2\pi)^d}\int d^dq \, \ln[M(q)]\right] = 0$$

$$= -i - \frac{1}{2}\frac{1}{(2\pi)^d}\int d^dq \, \frac{1}{M(q)}\frac{\partial M(q)}{\partial\lambda}$$

$$\sum_r e^{iq_1 r} M_{r,0}$$



$$M_{ab} = \beta J_{ab} + 2i\lambda\delta_{ab}$$

$$M(q) = \beta\left[(e^{iq_x a} + e^{-iq_x a}) + (e^{iq_y a} + e^{-iq_y a}) + \cdots\right]$$
$$+ 2i\lambda$$

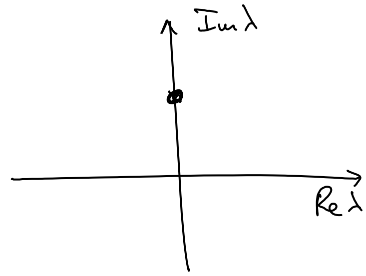
$$M(q) = 2\beta\sum_{j=1}^{d}\operatorname{Cos}(q_j a) = \beta J(q)$$

$$M(q) = \beta J(q) + 2i\lambda \qquad \frac{\partial M(q)}{\partial\lambda} = 2i$$

$$\frac{\partial[\ ]}{\partial\lambda} = 0 \iff \left[-i - \frac{i}{(2\pi)^d}\int d^dq \frac{1}{M(q)}\right] = 0$$

$$\boxed{-1 - \frac{1}{(2\pi)^d}\int d^dq \frac{1}{M^*(q)} = 0}$$

$$M(q) = \beta J(q) + 2i\lambda^*$$



$$-\beta\frac{\int\frac{d\lambda}{2\pi}\sum_{ab}(M^{-1})_{ab}\, e^{N\left[-i\lambda - \frac{1}{2N}\operatorname{Log}\operatorname{Det} M\right]}}{\int\frac{d\lambda}{2\pi}\, e^{N\left[-i\lambda - \frac{1}{2N}\operatorname{Log}\operatorname{Det} M\right]}} = -\frac{\beta\sum_{ab}(M^{-1}_{ab})_{\lambda^*} e^{N[\ \ ]_{\lambda^*}}}{e^{N[\ \ ]_{\lambda^*}}}$$

χ ... $\beta\sum(M^{-1})$ ... $\beta N\sum(M^{-1})$

$$\chi = -\beta \sum_{ab} (M^{-1})_{ab;\lambda^*} = -\beta N \sum_{a} (M^{-1}_{a,0})_{\lambda^*}$$

$$= -\beta N \quad M^{-1}(q=0)\Big|_{\lambda^*} = -\beta N \frac{1}{M(q=0)}\Big|_{\lambda^*}$$

$$\bar{\chi} = \chi / N$$

$$\begin{cases} \bar{\chi} = \dfrac{-\beta}{\beta\, J(q=0) + 2i\lambda^*} \\[2ex] 1 = -\dfrac{1}{(2\pi)^d}\displaystyle\int d^d q \; \dfrac{1}{\beta\, J(q) + 2i\lambda^*} \end{cases} \qquad J(q) = 2\sum_{j=1}^{d} \cos(q_j a)$$

$$\bar{\chi}^{-1} = -\left[J(q=0) + \frac{2i\lambda^*}{\beta}\right] \Rightarrow -\frac{2i\lambda^*}{\beta} = \bar{\chi}^{-1} + J(q=0)$$

$$1 = \frac{-1}{(2\pi)^d}\int d^d q \; \frac{1}{\beta J(q) - \beta\bar{\chi}^{-1} - \beta J(q=0)}$$

$$1 = \frac{-1}{(2\pi)^d}\int d^d q \; \frac{1}{\beta\left[J(q) - J(0)\right] - \beta\bar{\chi}^{-1}}$$

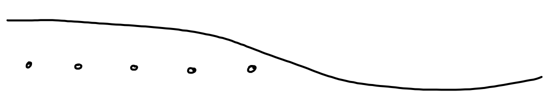
$$\boxed{\beta = \frac{1}{(2\pi)^d}\int d^d q \; \frac{1}{\left[J(0) - J(q)\right] + \bar{\chi}^{-1}}}$$

$$J(0) - J(q) = 2J\sum_{j=1}^{d}\left[1 - \cos(q_j a)\right] = 4J\sum_{j=1}^{d}\sin^2\left(\frac{q_j a}{2}\right)$$

$$-J\sum_{\langle ij\rangle} S_i S_j \qquad\qquad J(0) - J(q) \propto J q^2$$

$$-2S_i S_j \to (S_i - S_j)^2 - S_i^2 - S_j^2$$

$$\Delta E \simeq J\,(S_i - S_j)^2 \longrightarrow (\nabla S)^2$$



$q \ll 1/a \Rightarrow$ Continuum limit

$(\nabla S)^2 \to$ Ginzburg - Landau

\# <u>Existence of Tc.:</u> $\qquad \bar{\chi}^{-1} = 0 \qquad \bar{\chi} \to \infty$

$$\beta_c = \int \frac{d^dq}{(2\pi)^d} \frac{1}{J(0) - J(q)} \approx \int_0^{\Lambda} d^dq \, \frac{1}{q^2}$$

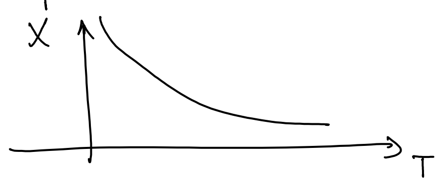
$\underline{d \leq 2}$ integral diverges $\beta_c = \infty \Rightarrow T_c = 0$

No ordered state

$$\beta = \int \frac{d^dq}{(2\pi)^d} \frac{1}{J(0) - J(q) + \chi^{-1}} = \int_0^{\Lambda} \frac{d^dq}{(2\pi)^d} \frac{1}{q^2 + \chi^{-1}}$$

$$\approx \int_{\sqrt{\chi^{-1}}}^{\Lambda} \frac{d^dq}{q^2} \sim q^{d-2} \sim (\chi^{-1})^{\frac{d-2}{2}} + \text{regular}$$

$$\beta \sim \chi^{\frac{2-d}{2}} \qquad \chi \sim \left(\frac{1}{T}\right)^{\frac{2}{2-d}}.$$



Mermin Wagner theorem. | Continuous Symmetry | no ordered state $(T \neq 0)$ $d \leq 2$

$(\nabla S)^2 \longrightarrow$ Goldstone modes.

\# $\underline{d > 2}$

$$\beta_c = \int \frac{d^dq}{(2\pi)} \frac{1}{J(0) - J(q)} \longrightarrow \text{finite}$$

$$\beta - \beta_c = \int \frac{d^dq}{(2\pi)^d} \left[\frac{1}{\Delta J + \chi^{-1}} - \frac{1}{\Delta J} \right]$$

$$= \int \frac{d^dq}{(2\pi)^d} \left[\frac{-\chi^{-1}}{(\Delta J)\left[\Delta J + \chi^{-1}\right]} \right]$$

$$\beta_c - \beta = \chi^{-1} \int \frac{d^dq}{(2\pi)^d} \frac{1}{\Delta J \left[\Delta J + \chi^{-1}\right]}$$

if I can expand $\beta_c - \beta = \chi^{-1}\,\text{Cste} \Rightarrow \chi^{-1} = \frac{1}{T_c} - \frac{1}{T}$

$\chi \propto \frac{1}{T_c - T}$ Ginzburg - Landau result $\gamma = 1$

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{[\Delta J]^2} \qquad \int \frac{d^d q}{q^4}$$

$\underline{d > 4}$ one can expand.

$\longrightarrow$ recovers the GL exponent in $\underline{\text{all}}$ dimension.

GL theory becomes "exact"

$\underline{d \leq 4}$

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 (q^2 + \chi^{-1})} \simeq \int_{\sqrt{\chi^{-1}}}^{\Lambda} \frac{d^d q}{q^4} \sim q^{d-4}$$

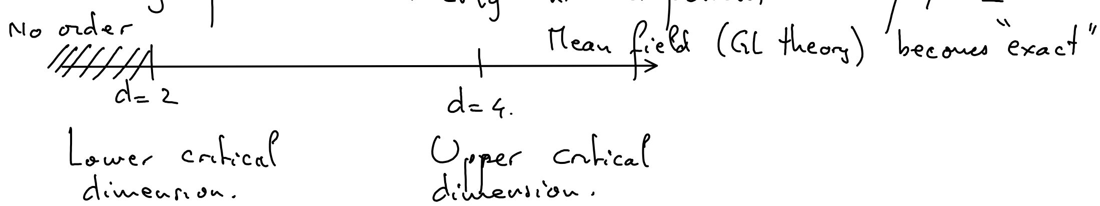
$$\sim (\chi^{-1})^{\frac{d-4}{2}}$$

$$\beta_c - \beta = \chi^{-1} \cdot (\chi^{-1})^{\frac{d-4}{2}} = (\chi^{-1})^{\frac{d-2}{2}}$$

$$\frac{1}{T_c} - \frac{1}{T} = \frac{T - T_c}{T_c T} \simeq \frac{T - T_c}{T_c^2} \propto (\chi^{-1})^{\frac{d-2}{2}}$$

$$\chi \sim \left(\frac{T_c^2}{T - T_c} \right)^{\frac{2}{d-2}}$$

GL theory predicts incorrectly the exponent. $\gamma \neq 1$



Exercise

① $H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$ N component Ising model

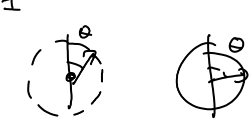
$$S = (S_1, S_2, S_3, \dots S_N) \qquad \sum_{\alpha=1}^{N} S_\alpha^2 = 1$$

$S_\alpha \in]-\infty, +\infty[$

$N = 1$ Ising model.

$N = 2$ $\quad S_x, S_y \quad S_x^2 + S_y^2 = 1$

$\vec{S} = (\cos\theta, \sin\theta)$



$$H = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

XY Model

XY Model. Solve the model for $N \to \infty$

② Ising. $H = -J \sum_{\langle ij \rangle} \sigma_i \cdot \sigma_j$. $\sigma_i = \pm 1$.

$$Z = \sum_{\{\sigma\}} e^{-\beta H[\sigma]}$$

field $S_i \in]-\infty, +\infty[$ on each site

$$Z = \sum_{\{\sigma\}} \int \mathcal{D} S_i \; e^{-\beta H[S]} \prod_i \delta(S_i - \sigma_i)$$

- express the constraint in exponential form.
- Sum explicitly over the σ_i → Get "H" for the fields S_i and the fields associated with the constraint.
- Do a saddle point approximation. What is the physics.

II] Perturbation theory.

$$F[\phi] = \int dx \left[a(T)\phi^2 + b\phi^4 + \frac{c}{2}(\nabla\phi)^2 \right]$$

$$a(T) = \alpha (T - T_c)$$

$$Z = \int \mathcal{D}\phi \; e^{-F[\phi]} \longrightarrow \int \mathcal{D}\phi \; e^{-NF[\phi]}$$

$$Z = \int \mathcal{D}\phi \; e^{-\int dx \left[\frac{1}{2} a(T) \phi^2 + \frac{c}{2}(\nabla\phi)^2 + b\phi^4(x) \right]}$$

$b = 0$ $$Z_0 = \int \mathcal{D}\phi \; e^{-\int dx \left[\frac{1}{2} a(T) \phi^2 + \frac{c}{2}(\nabla\phi)^2 \right]}$$

$$= \int \mathcal{D}\phi_q \mathcal{D}\phi_q^* \; e^{-\frac{1}{\Omega}\sum_q \left[\frac{1}{2} a(T) \phi_q^* \phi_q + \frac{c}{2} q^2 \phi_q^* \phi_q \right]}$$

expand in the non-Gaussian part of the integral.

$$Z = \int \mathcal{D}\phi \; e^{-\int dx \left[\frac{1}{2} a(T) \phi^2 + \frac{1}{2} c (\nabla\phi)^2 \right]} \; e^{-\int dx \, b \phi^4}$$

$$\left[1 - \int dx \, b\phi^4 + \frac{1}{2}(----)^2 \cdots \right.$$

$$Z = Z_0 \left[1 - \frac{b \int dx_0 \int \mathcal{D}\phi \; \phi^4(x_0) \; e^{-\int dx [\quad]\phi\phi}}{\int \mathcal{D}\phi \; e^{-\int dx [\quad]\phi\phi}} + \cdots \right.$$

$$\frac{1}{2} b^2 \int dx_0 \int dx_1 \quad \frac{\int \mathcal{D}\phi \quad \phi^4(x_0)\, \phi^4(x_1) \quad e^{-(\ )}}{\int \mathcal{D}\phi \quad e^{-(\ )}} \quad + \ldots$$

$$\langle \phi(x_0)\, \phi(x_1)\, \phi(x_2)\, \phi(x_3) \ldots\ldots \phi(x_n) \rangle_{H_0}$$

H_0 is a quadratic Hamiltonian.

$$\beta H_0 = \frac{1}{2} \int dx_1\, dx_2 \quad \phi(x_1)\, G^{-1}(x_1 - x_2)\, \phi(x_2)$$

$$= \frac{1}{2\Omega} \sum_q G^{-1}(q) \quad \phi^*(q)\, \phi(q).$$

<u>example.</u> GL. functional. $\int dx \ \frac{1}{2} a(T)\, \phi^2 + \frac{c}{2} (\nabla \phi)^2$

$$\rightarrow \frac{1}{2\Sigma} \sum_q \left[a(T) + cq^2 \right] \phi_q^* \phi_q.$$

$$\langle u(x_1)\, u(x_2) \rangle = G(x_1 - x_2)$$

$$\int dx_3 \ G(x_1 - x_3)\, G^{-1}(x_3 - x_2) = \delta(x_1 - x_2)$$

$$\beta H = \frac{1}{2} \left[\int dx_1\, dx_2 \quad \phi(x_1)\, G^{-1}(x_1 - x_2)\, \phi(x_2) \quad + 2 \int dx \ h(x)\, \phi(x) \right]$$

$$Z = \int \mathcal{D}\phi \quad e^{-\beta H}. \qquad\qquad F = \frac{-1}{\beta} \operatorname{Log}[Z]$$

$$\langle \phi(x_1)\, \phi(x_2)\, \phi(x_3)\, \phi(x_4) \ldots \rangle$$

$$\frac{\partial}{\partial h(x_1)} \frac{\partial}{\partial h(x_2)} \frac{\partial}{\partial h(x_3)} \quad e^{-\beta H} \Big|_{h=0} = \phi(x_1)\, \phi(x_2)\, \phi(x_3)\, e^{-\beta H}. \Big|_{h=0}$$

$$= \phi(x_1)\, \phi(x_2)\, \phi(x_3) \ e^{-\beta H_0}.$$

$$\frac{\int \mathcal{D}\phi \quad \phi(x_1)\, \phi(x_2) \ldots e^{-\beta H_0}.}{\int \mathcal{D}\phi \quad e^{-\beta H_0}} = \frac{\partial}{\partial h(x_1)} \frac{\partial}{\partial h(x_2)} \frac{\partial}{\partial h(x_3)} \frac{\int \mathcal{D}\phi \quad e^{-\beta H}}{\int \mathcal{D}\phi \quad e^{-\beta H_0}}$$

$$\beta H = \frac{1}{2} \int dx_1\, dx_2 \quad \phi(x_1)\, G^{-1}(x_1 - x_2)\, \phi(x_2) \quad + 2 \int h(x)\, \phi(x)$$

$$= \frac{1}{2} \int dx_1 \int dx_2 \; [\underbrace{\phi(x_1) + \int dx_3 \, G(x_1 - x_3) h(x_3)}_{\tilde{\Phi}(x)}] \, G^{-1}(x_1 - x_2) \, [\phi(x_2) + \int dx_3 \, G(x_2 - x_3) h(x_3)]$$

$$- \frac{1}{2} \int dx_1 dx_2 \; h(x_1) \, G(x_1 - x_2) \, h(x_2)$$

$$\frac{\int \mathcal{D}\phi \; e^{-\beta H}}{\int \mathcal{D}\phi \; e^{-\beta H_0}} = \frac{\int \mathcal{D}\tilde{\phi} \; e^{-\frac{1}{2}\int \tilde{\Phi}(x_1) G^{-1}(x_1 - x_2) \tilde{\phi}(x_2)}}{\int \mathcal{D}\phi \; e^{-\frac{1}{2}\int \phi(x_1) G^{-1}(x_1 - x_2) \phi(x_2)}} \; e^{+\frac{1}{2}\int h(x_1) G(x_1 - x_2) h(x_2)}$$

$$= e^{\frac{1}{2} \int dx_1 dx_2 \; h(x_1) \, G(x_1 - x_2) \, h(x_2)} = A.$$

$$\frac{\partial}{\partial h(x_1^0)} \frac{\partial}{\partial h(x_2^0)} A \Big|_{h=0} = \frac{\partial}{\partial h(x_2^0)} \left[\frac{1}{2} \int G(x_1^0 - x_2) \, h(x_2) \, dx_2 + \frac{1}{2} \int dx_1 \, G(x_1 - x_1^0) \, h(x_1) \right] A$$

$$= \frac{\partial}{\partial h(x_2^0)} \left[\int G(x_1^0 - x_2) \, h(x_2) \, dx_2 \quad A \right]$$

$$= G(x_1^0 - x_2^0) \, A + \left(\int G(\;) h \right) \left(\int G \, h \right) A$$

$$\xrightarrow[h \to 0]{} G(x_1^0 - x_2^0)$$

$$\langle \phi(x_1) \, \phi(x_2) \, \phi(x_3) \, \phi(x_4) \rangle = \frac{\partial}{\partial h_1 \partial h_2 \partial h_3 \partial h_4}$$

Sum of all possible pairing of the fields.

$$= \underbrace{G(x_1 - x_2)}_{\langle \phi_1 \phi_2 \rangle} \underbrace{G(x_3 - x_4)}_{\langle \phi_3 \phi_4 \rangle} + \underbrace{G(x_1 - x_3)}_{\langle \phi_1 \phi_3 \rangle} \underbrace{G(x_2 - x_4)}_{\langle \phi_2 \phi_4 \rangle}$$

$$+ \underbrace{G(x_1 - x_4)}_{\langle \phi_1 \phi_4 \rangle} \underbrace{G(x_2 - x_3)}_{\langle \phi_2 \phi_3 \rangle}$$

Wick theorem. General property of Gaussian integrals.

$$\langle \phi^4(x_0) \rangle = \langle \phi(x_0) \, \phi(x_0) \, \phi(x_0) \, \phi(x_0) \rangle$$

$$= 3 \, G(0) \, G(0)$$

$$G(x=0) = \sum_q e^{iq(x=0)} \, G(q)$$

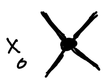
$$GL \rightarrow \frac{1}{2\sum} \sum_q \left[a(T) + cq^2 \right] \phi_q^* \phi_q \Rightarrow G^{-1}(q) = a(T) + cq^2.$$

$$G(q) = \frac{1}{G^{-1}(q)} \quad \text{(diagonal)} \qquad G(q) = \frac{1}{a(T) + cq^2}.$$

$$\langle \phi^4(x_0) \rangle = 3\left[\sum_q \frac{1}{a(T) + cq^2} \right]$$

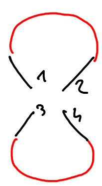
Feynman diagrams.

$$\phi^4(x_0) \longrightarrow \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)$$

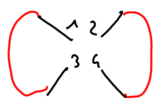


$$\langle \phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4) \rangle$$

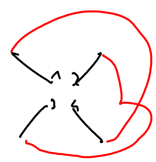
all the connections of the lines.



$G(x_1 - x_2)\, G(x_3 - x_4)$



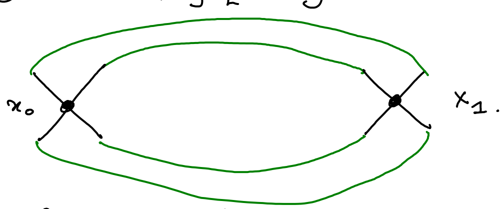
$G(x_1 - x_3)\, G(x_2 - x_4)$



$G(x_1 - x_4)\, G(x_2 - x_3)$

2^{nd} order.

$$\frac{1}{2}\left[b \int dx_0\, \phi^4(x_0) \right]\left[b \int dx_1\, \phi^4(x_1) \right]$$



one pairing.

$$\frac{b^2}{2} \int dx_0 \int dx_1 \quad G^4(x_0 - x_1)$$

$$\frac{b^2}{2} \int dx_0 \int dx_1 \quad G^2(x_0 - x_1)\, G^2(0)$$