Note that there seems to be some arbitrariness in the above expressions in terms of the bosonic fields since by anticommuting two fermionic fields one can introduce a minus sine and thus change a sine into a cosine. We will see in Section 4.3.2 how to answer this question in an unambiguous way. The decay of these correlation functions is trivially obtained from (2.114) by letting $k_F = 0$ and $K_\rho \to 1/K_\rho$. Here again this is a power law decay. There is no true superconducting *order* (since $\langle O \rangle = 0$) due to the impossibility to break a continuous symmetry in one dimension. The best the system can do is to have slowly (power law) decaying correlations.

Having the correlation functions we could start building the phase diagram in the same way like that for spinless fermions. This would be valid for a system with $g_{1\perp} = 0$. However, the effects of the $g_{1\perp}$ terms are quite drastic so it will be good to understand them first. This is of course mandatory if we want to be able to deal with spin rotation invariant Hamiltonians. It is also a quite important calculation since sine-Gordon-type Hamiltonians are common in the one-dimensional world and we will have to learn how to deal with them in more complicated situations as well.

2.3.2 *Renormalization equations for sine-Gordon Hamiltonians*

To complete our analysis of the spin sector we have to treat the sine-Gordon Hamiltonian (2.106). There are many ways one can tackle such a problem. The fact that interactions (or other terms) can generate non-quadratic terms is a recurrent fact when dealing with one-dimensional systems. This is obvious from the exponential form of the single fermion operators. When putting together a certain number of these operators, quite generally exponentials of the boson fields will remain. Physically, the effect of the cosine is clear. Contrary to the quadratic term that lets the field ϕ fluctuate, the cosine term would like to lock the field ϕ in one of the minima of the cosine. There will thus be a competition between the quadratic part and the cosine. Since the quadratic part contains the conjugate momentum Π it does not like ϕ to be blocked and promotes fluctuations. In order to know who wins and to obtain the low-energy physical properties of (2.106) we use a renormalization procedure. The ideas of a renormalization procedure are explained in detail in Section 1.3.2 and I strongly encourage you to read this section to understand the philosophy of the procedure, before we start with the gory details of the calculation itself.

As discussed in Section 1.3.2 we change the cutoff of the system while keeping the low-energy properties of the system unchanged. This can be done by varying the coupling constants (and possibly generating new couplings). There are various ways to carry this renormalization procedure. I give in this section a derivation based directly on the correlation functions (José *et al.*, 1977). It has the advantage of being physically transparent and to extend easily to more complicated cases (see Section 4.4). An alternative derivation directly on the partition function is given in Appendix E.1.

Let us consider the correlation function

$$R(r_1 - r_2) = \langle e^{ia\sqrt{2}\phi(r_1)} e^{-ia\sqrt{2}\phi(r_2)} \rangle_H \tag{2.117}$$

If H was the quadratic Hamiltonian H^0 of (2.41), the correlation would be (for $r_1 - r_2 \gg \alpha$)

$$\langle e^{ia\sqrt{2}\phi(r_1)} e^{-ia\sqrt{2}\phi(r_2)} \rangle_{H^0} = e^{-a^2 K F_1(r_1 - r_2)} \simeq \left(\frac{\alpha}{r_1 - r_2} \right)^{a^2 K} \tag{2.118}$$

as shown in Appendix C. Since (2.106) contains a cosine, one cannot compute (2.117) exactly any more, but one can perform a perturbative expansion in the cosine term, assuming that the interaction $g_{1\perp}$ is small. In the following, to lighten the notations I will denote $g_{1\perp}$ simply by g. The first-order term is zero, and if we stop at second order the correlation function is given by

$$\begin{aligned} R(r_1 - r_2) = \langle e^{ia\sqrt{2}\phi(r_1)} e^{-ia\sqrt{2}\phi(r_2)} \rangle_{H_0} + \frac{1}{2} \left(\frac{g}{(2\pi\alpha)^2 u} \right)^2 \sum_{\epsilon_1 = \pm 1, \epsilon_2 = \pm 1} \\ \int\int d^2r'\, d^2r'' [\langle e^{ia\sqrt{2}\phi(r_1)} e^{-ia\sqrt{2}\phi(r_2)} e^{i\epsilon_1\sqrt{8}\phi(r')} e^{-i\epsilon_2\sqrt{8}\phi(r'')} \rangle_{H_0} \\ - \langle e^{ia\sqrt{2}\phi(r_1)} e^{-ia\sqrt{2}\phi(r_2)} \rangle_{H_0} \langle e^{i\epsilon_1\sqrt{8}\phi(r')} e^{-i\epsilon_2\sqrt{8}\phi(r'')} \rangle_{H_0}] \end{aligned} \tag{2.119}$$

where $r = (x, y = u\tau)$ and $d^2r = dx\, dy$. The second term in the integral is the disconnected terms coming from the partition function in the denominator of the average. Since the averages are taken with the quadratic Hamiltonian (2.41) they can be readily performed (see Appendix C). One gets

$$\begin{aligned} R(r_1 - r_2) = e^{-a^2 K F_1(r_1 - r_2)} \left[1 + \frac{g^2}{2(2\pi\alpha)^4 u^2} \sum_{\epsilon_1} \int\int d^2r' d^2r'' e^{-4KF_1(r' - r'')} \right. \\ \left. \left(e^{2a\epsilon_1 K[F_1(r_1 - r') - F_1(r_1 - r'') + F_1(r_2 - r'') - F_1(r_2 - r')]} - 1 \right) \right] \end{aligned} \tag{2.120}$$

The terms with $\epsilon_2 \neq \epsilon_1$ vanish (see Appendix C). We can take a as we wish. It is clear that if one takes a small enough, due to the factor $e^{-4F(r' - r'')}$ (which is essentially a power law) the integral over $d^2r'\, d^2r''$ is dominated by configurations where r' and r'' are not too distant from each other. One can thus expand the exponential term linear in a in powers of $r' - r''$. In particular, if we introduce the center of mass and relative coordinates

$$\begin{aligned} R &= \frac{r' + r''}{2} \\ r &= r' - r'' \end{aligned} \tag{2.121}$$

we can rewrite the correlation function as

$$R(r_1 - r_2) = e^{-a^2KF_1(r_1-r_2)}\left[1 + \frac{g^2}{2(2\pi\alpha)^4u^2}\sum_{\epsilon_1}\int\int d^2R\,d^2r\,e^{-4KF_1(r)}\right.$$
$$\left.\left(e^{2a\epsilon_1 K[r\cdot\nabla[F_1(r_1-R)-F_1(r_2-R)]}-1\right)\right] \quad (2.122)$$

Since r is small the exponential can be expanded. The first-order term is zero because of the sum over ϵ_1. Stopping at second order gives for the correlation function

$$R(r_1 - r_2) = e^{-a^2KF_1(r_1-r_2)}\left[1 + \frac{2g^2}{(2\pi\alpha)^4u^2}\int\int d^2Rd^2r\,e^{-4KF_1(r)}\right.$$
$$\left.(aK[r\cdot\nabla[F_1(r_1-R)-F_1(r_2-R)])^2\right] \quad (2.123)$$

The expansion of the quadratic term leads to terms of the form

$$r_ir_j(\nabla_{R_i}[F_1(r_1-R)-F_1(r_2-R)])(\nabla_{R_j}[F_1(r_1-R)-F_1(r_2-R)]) \quad (2.124)$$

where i,j denote the two possible coordinates x and y. Let us make use of the rotation invariance in $(x,u\tau)$ of the quadratic Hamiltonian and choose a cutoff procedure that respects this invariance. The asymptotic properties are independent of the short distance cutoff so this choice is arbitrary. Such a cutoff procedure corresponds to restricting $r > \alpha$. In that case it is easy to see that because of the integral over d^2r in (2.123) only the terms with $i = j$ survive, the other terms being zero by symmetry $x \to -x$ or $y \to -y$. Since moreover $\int d^2r\,x^2 = \int d^2r\,y^2 = \int d^2r\,r^2/2$, one obtains by integration by part over R

$$R(r_1 - r_2) = e^{-a^2KF_1(r_1-r_2)}\left[1 - \frac{g^2}{(2\pi\alpha)^4u^2}\int\int d^2R\,d^2r\,e^{-4KF_1(r)}a^2K^2r^2\right.$$
$$\left.[F_1(r_1-R)-F_1(r_2-R)](\nabla_X^2+\nabla_Y^2)[F_1(r_1-R)-F_1(r_2-R)]\right] \quad (2.125)$$

Since $F_1(r)$ is essentially a log ($F_1(r) = \log(r/\alpha)$ when $r \gg \alpha$) one can use

$$(\nabla_X^2 + \nabla_Y^2)\log(R) = 2\pi\delta(R) \quad (2.126)$$

This identity, well-known for two-dimensional Coulomb systems, can be directly proven by differentiating (C.25). More generally even with an isotropic cutoff

$$\int d^2q[1-\cos(qr)]\frac{1}{q^2}e^{-\alpha q} = 2\pi\log\left[\frac{1}{2}(1+\sqrt{1+r^2/\alpha^2})\right] \simeq 2\pi\log[r/\alpha] \quad (2.127)$$

Applying $(\nabla_X^2 + \nabla_Y^2)$ on (2.127) obviously gives back (2.126). I will come back to the analogy between this problem and two-dimensional Coulomb problems in Section 3.3. Using (2.126) in (2.125) one obtains

$$\int d^2R[F_1(r_1-R)-F_1(r_2-R)](\nabla_X^2+\nabla_Y^2)[F_1(r_1-R)-F_1(r_2-R)] = -4\pi F_1(r_1-r_2) \tag{2.128}$$

The terms $F_1(r_1-r_1)$ are finite (not logarithmically divergent) because the correlation function F_1 is regular at a short distance. With our original regularization scheme (2.64) this term would be zero. The term $F_1(r_1-r_2)$ is logarithmically divergent when the distance between r_1 and r_2 becomes large. The correlation function is thus

$$R(r_1-r_2) = e^{-a^2KF_1(r_1-r_2)}\left[1+\frac{g^2K^2a^2F_1(r_1-r_2)}{4\pi^3u^2\alpha^4}\int_{r>\alpha} d^2r\, r^2e^{-4F_1(r)}\right] \tag{2.129}$$

where $d^2r = dx\, dy = 2\pi r\, dr$. One recognizes an expansion of an exponential form similar to (2.118) but with an effective exponent K_{eff}

$$K_{\text{eff}} = K - \frac{y^2K^2}{2}\int_\alpha^\infty \frac{dr}{\alpha}\left(\frac{r}{\alpha}\right)^{3-4K} \tag{2.130}$$

where I have used $y = g/(\pi u)$. The exponent of the correlation function is precisely what controls the asymptotic (low-energy) properties of the system. This exponent should remain unaffected by the cutoff. If we vary the cutoff from α to $\alpha' = \alpha + d\alpha$ in the limit of the integral one has

$$K_{\text{eff}} = K - \frac{y^2K^2}{2}\frac{d\alpha}{\alpha} - \frac{y^2K^2}{2}\int_{\alpha'}^\infty \frac{dr}{\alpha}\left(\frac{r}{\alpha}\right)^{3-4K} \tag{2.131}$$

To keep K_{eff} unchanged one should change the parameter K such that

$$K(\alpha') = K(\alpha) - \frac{y^2(\alpha)K^2(\alpha)}{2}\frac{d\alpha}{\alpha} \tag{2.132}$$

Similarly, to get back (2.130) but with α' one should rescale the integral and thus define

$$y^2(\alpha') = y^2(\alpha)\left(\frac{\alpha'}{\alpha}\right)^{4-4K(\alpha)} \tag{2.133}$$

The forms of (2.132) and (2.133) suggest to parametrize $\alpha = \alpha_0 e^l$ where α_0 is the original cutoff. Changing α is thus equivalent to change l into $l + dl$. Using this parametrization in (2.132) and (2.133) and making an infinitesimal change gives the renormalization equations

$$\begin{aligned} \frac{dK(l)}{dl} &= -\frac{y^2(l)K^2(l)}{2} \\ \frac{dy(l)}{dl} &= (2-2K(l))y(l) \end{aligned} \tag{2.134}$$

Note that these equations are only perturbative in y but are exact in K.

Before we analyze these equations let us understand them on a physical basis. The equation for y is simply the scaling dimension of the operator that appears in the action

$$y \int dx \int d\tau \ \cos(\sqrt{8}\phi_\sigma) \tag{2.135}$$

Since the correlation behaves as

$$\langle \cos(\sqrt{8}\phi_\sigma(r)) \ \cos(\sqrt{8}\phi_\sigma(0)) \rangle = \left(\frac{\alpha}{r}\right)^{4K} \tag{2.136}$$

we can say that the operator behaves as

$$\cos(\sqrt{8}\phi_\sigma) \equiv L^{-2K} \tag{2.137}$$

Thus, the operator in the action behaves as

$$y \int dx \int d\tau \ \cos(\sqrt{8}\phi_\sigma) \equiv y L^{2-2K} \tag{2.138}$$

which gives back the equation for y. One can understand the equation for K by noting that K controls the fluctuations of ϕ_σ through the term

$$\frac{u}{K}(\nabla\phi_\sigma)^2 \tag{2.139}$$

in the Hamiltonian. Since the cosine term wants to order the field ϕ_σ its effect at a purely quadratic level would be to decrease K to make the fluctuations of ϕ_σ more difficult in (2.139). A similar physical picture is given by the Wilson renormalization of Appendix E.1.

The flow (2.134) is shown in Fig. 2.6. For an infinitesimal y, one sees from (2.134) that for $K < 1$ y is relevant (y grows upon a change of scale), whereas for $K > 1$ y is irrelevant (y decreases upon a change of scale). One thus expects a phase transition at $K = 1$. To further analyze the flow around this transition let us expand $K_\rho = 1 + y_\parallel/2$. Note that this corresponds to the original $y_{1\parallel}$ and $y_{1\perp}$ from (2.105). With these variables the flow becomes

$$\begin{aligned} \frac{dy_\parallel(l)}{dl} &= -y^2(l) \\ \frac{dy(l)}{dl} &= -y_\parallel(l)y(l) \end{aligned} \tag{2.140}$$

These equations that correspond to an expansion up to second order in the interactions are identical to the ones derived in the fermion language (1.58). They are also identical to the ones derived for the XY problem (Kosterlitz, 1974), as we will discuss more in Section 3.1. From (2.140) we see that $y\frac{dy}{dl} = y_\parallel \frac{dy_\parallel}{dl}$ and thus

$$A^2 = y_\parallel^2 - y^2 \tag{2.141}$$

is a constant of motion. The trajectories are thus hyperbolas. A real corresponds to the regions (a) and (c) in Fig. 2.6. A imaginary ($A^2 < 0$) is region (b). The

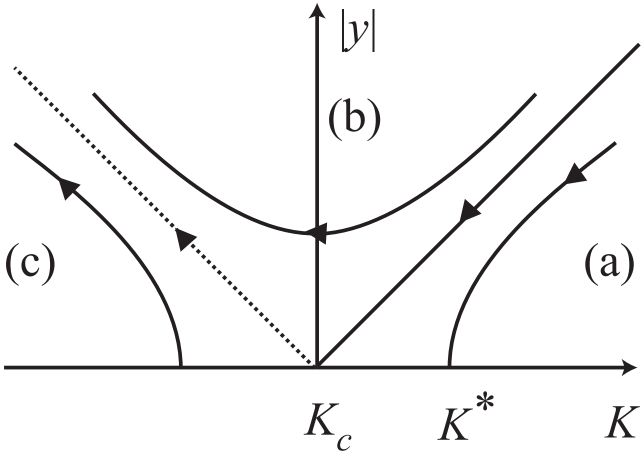

FIG. 2.6. Flow corresponding to the eqn (2.134). The trajectories are hyperbolas (see text). The first diagonal (thick line) is a separatrix between a regime where y is irrelevant and a regime where y flows to strong coupling.

equations depend only on $|y|$ and thus the flow is symmetric when changing $y \to -y$. It will be thus sufficient to analyze the case $y > 0$.

The line $y_\| = y$ is obviously a separatrix between two different regimes, as shown in Fig. 2.6. Using the constant of motion (2.141) in (2.140) the flow can be easily integrated. For example, for $y_\| > y$ one has

$$\begin{aligned} y_\|(l) &= A/(\tanh(Al + \operatorname{atanh}(A/y_\|^0))) \\ y(l) &= A/(\sinh(Al + \operatorname{atanh}(A/y_\|^0))) \end{aligned} \tag{2.142}$$

On the separatrix one has

$$y_\|(l) = y(l) = \frac{y^0}{1 + y^0 l} \tag{2.143}$$

When $A > 0$ and $y_\| > 0$, the operator $\cos(\sqrt{8}\phi_\sigma)$ is irrelevant. The fixed point corresponds to $y^* = 0$ and $y_\|^* = A$. Close to the fixed point the flow can be approximated by

$$\begin{aligned} \frac{dy_\|(l)}{dl} &= 0 \\ \frac{dy(l)}{dl} &= (2 - 2K^*)y(l) \end{aligned} \tag{2.144}$$

To obtain the correlation functions we should, strictly speaking, write the renormalization equations for the correlation functions themselves. This will be done in Section 4.4. However, the trajectories are nearly vertical, thus $y \to 0$ while K converges to a fixed point value K^*. As a first approximation one can thus compute the correlation functions using the simple quadratic Hamiltonian (2.41) but with the renormalized parameters K^*. As a result the correlations have the asymptotic decay

$$\langle e^{ia\sqrt{2}\phi_\sigma(r)} e^{-ia\sqrt{2}\phi_\sigma(0)} \rangle \simeq \left(\frac{\alpha}{r}\right)^{a^2 K_{\text{eff}}} = \left(\frac{\alpha}{r}\right)^{a^2 K^*} \tag{2.145}$$

In this regime the RG flow thus allows to get the asymptotic behavior of the correlation functions. Because the cosine disappears from the asymptotic properties and the system is described by a pure quadratic Hamiltonian, this regime is called 'massless regime'.

When $y_\parallel = y$ the operator is marginally irrelevant, the flow is along the first diagonal. We thus see that if one starts from a spin rotation invariant problem $y = y_\parallel$ then the rotation invariance is preserved at each step of the renormalization as it should be. The fixed point corresponds to $y_\parallel^* = y^* = 0$ and thus $K^* = 1$. One can thus still use the expression (2.145), with $K^* = 1$. As we discussed in the previous section this is exactly the value that ensures spin rotation invariance of the correlation functions, which now is only natural. However, the fact that the corresponding operator is only marginal gives some additional contributions as shown in Section 4.4. For some correlation functions this gives rise to logarithmic corrections. For example,

$$\begin{aligned} \langle \cos(\sqrt{2}\phi_\sigma(r)) \ \cos(\sqrt{2}\phi_\sigma(0)) \rangle &\simeq \frac{\alpha}{r} \log^{1/2}(r/\alpha) \\ \langle \sin(\sqrt{2}\phi_\sigma(r)) \ \sin(\sqrt{2}\phi_\sigma(0)) \rangle &\simeq \frac{\alpha}{r} \log^{-3/2}(r/\alpha) \end{aligned} \tag{2.146}$$

See Section 4.4 for the general expressions.

When $y > y_\parallel$ the trajectories tend to $y_\parallel \to -\infty$ and $y \to \infty$. The flow goes to strong coupling. Of course, since the RG equations themselves have been established in a perturbation expansion in y they cease to be valid beyond a certain lengthscale for which $y(l) \sim 1$. Nevertheless, the equations can be used below this lengthscale. The flow can be integrated as

$$\arctan(y_\parallel^0/\overline{A}) - \arctan(y_\parallel/\overline{A}) = \overline{A} l \tag{2.147}$$

where

$$\overline{A} = \sqrt{y_0^2 - (y_\parallel^0)^2} \tag{2.148}$$

On the special line $y_\parallel = -y < 0$ (for which $\overline{A} = 0$) one has

$$y(l) = \frac{y^0}{1 - y^0 l} \tag{2.149}$$

note that this line corresponds also to the spin rotation invariant case since one can take $y = -y_{1\perp}$ when $y_{1\parallel} = y_{1\perp} < 0$. Since the flow goes to strong coupling we need to guess the physics of this phase. This is one major advantage of the boson representation over the fermion one. We can try to analyze the Hamiltonian by looking at the limit $y \to \pm\infty$ (and $K_\sigma \to 0$). On the fermion representation having a coupling constant going to infinity does not help much, because fermions operators do not have a classical limit. On the boson Hamiltonian we can expect that when $y \to \pm\infty$ the term

$$\frac{yu}{2\pi\alpha^2}\int dx\ \cos(\sqrt{8}\phi_\sigma) \tag{2.150}$$

imposes that ϕ_σ is locked into one of the minima of the cosine. The field ϕ_σ thus orders and we go into a massive phase. If y is very large one can expand the cosine around the minimum. If $y \to -\infty$ the minimum is $\phi_\sigma = 0$ and the Hamiltonian would become

$$H = H^0 + \frac{2yu}{\pi\alpha^2}\int dx\ \phi_\sigma^2(x) \tag{2.151}$$

In Fourier space the action would thus become

$$S = \frac{1}{2\pi K}\frac{1}{\beta\Omega}\sum_{k,\omega_n}\left[\frac{1}{u}\omega_n^2 + uk^2 + \frac{4Kyu}{\alpha^2}\right]\phi^*(k,\omega_n)\phi(k,\omega_n) \tag{2.152}$$

Even at $k = 0$ excitations now cost a finite energy. The spectrum thus has a gap of order

$$M_P = \sqrt{\frac{4Kyu^2}{\alpha^2}} \tag{2.153}$$

These excitations are the 'phonons' (that is, the small oscillations) of the field ϕ which are now massive. They correspond to a variation of ϕ within one of the minima of the cosine. There are other excitations, the solitons, that take ϕ from one minimum of the cosine to the other. I will come back to such solutions later (see also Appendix E.3). Of course, such an expansion is valid only if y is very large. One can make a more sophisticated approximation using a variational approach as shown in Appendix E.2. A more accurate method is to combine the RG with a strong coupling analysis. The boson Hamiltonian is quite useful in that respect since, contrary to the fermion Hamiltonian, it is easy to analyze the limit where the coefficient of the cosine becomes large. Let us use the RG up to a point where the coupling y is of order one. The gap in the spectrum has the dimension of an energy and thus it renormalizes as

$$\Delta_\sigma(l) = e^l\Delta_\sigma(l=0) \tag{2.154}$$

When $y(l^*) \sim 1$ we can use the expansion (2.152), which would lead to a gap $\Delta_\sigma(l^*) \sim uy^{1/2}(l^*)/\alpha \sim u/\alpha$. The true gap of the system is thus simply given by

$$\Delta_\sigma(l=0) \simeq e^{-l^*}\Delta_0 \tag{2.155}$$

where l^* is the scale at which the coupling is of order one and $\Delta_0 = u/\alpha$ is a quantity of the order of the original bandwidth of the system. The complete solution can easily be obtained from the flow (2.147). Let us examine it in three physically different limits.

$y \ll |y_{\|}|$. In that case one is deep in the massive phase. From Fig. 2.6 and the flow (2.147), one sees that the flow is nearly vertical. One can thus approximate it by

$$\begin{aligned} \frac{dK(l)}{dl} &= 0 \\ \frac{dy(l)}{dl} &= (2-2K)y(l) \end{aligned} \tag{2.156}$$

which gives $y(l) = y^0 e^{(2-2K)l}$. Thus, the lengthscale l^* is

$$e^{l^*} = \left(\frac{1}{y^0}\right)^{1/(2-2K)} \tag{2.157}$$

and the gap is

$$\Delta_\sigma(l=0)/\Delta_0 \simeq (y^0)^{1/(2-2K)} \tag{2.158}$$

The gap is thus a power law of the bare coupling constant, with an exponent controlled by K. This result can also be obtained by a variational approach (see Appendix E.2). If one gets closer to the transition $K \to 1$ the gap gets smaller as it should (remember that $y^0 \ll 1$).

$y_{\|} = -y$. This corresponds to a system which is invariant by spin rotation. In that case, (2.149) gives (for $y \ll 1$)

$$l^* = \frac{1}{y^0} - 1 \sim \frac{1}{y^0} \tag{2.159}$$

The gap is thus

$$\Delta_\sigma(l=0) \simeq \Delta_0 e^{-1/y^0} \tag{2.160}$$

and the gap is *exponentially* small in the coupling constant. This is clearly a highly non-perturbative result. I will come back to this result when discussing the Hubbard model in Section 7.1.1.

Close to the Transition. Finally let us get close to the separatrix. In that case using (2.147) gives

$$\overline{A} l^* \to \pi \tag{2.161}$$

since $\overline{A} \to 0$ close to the transition and thus $y_{\|}^0/\overline{A} \to \infty$ and $y_{\|}(l^*)/\overline{A} \to -\infty$. Thus, the gap becomes

$$\Delta_\sigma(l=0) \simeq \Delta_0 e^{-\pi/\overline{A}} \tag{2.162}$$

If we consider that one approaches the transition such as $y_0 \to y_{\|}$ then

$$\overline{A} \sim \sqrt{2y_0(y_0 - y_{\|}^0)} \tag{2.163}$$

and thus the gap is exponentially small in the square root of the distance to the transition.