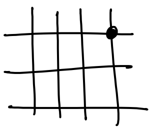
Equilibrium Statistical Mecanics.

I.] Reminders, Correlation functions. :



$\sigma_i = \pm 1.$ $\qquad C = \{\sigma_1 \text{------} \sigma_N\}$

$H[\{\sigma\}]$

$$H = -J \sum_{\langle ij \rangle} \sigma_i \cdot \sigma_j$$

$\uparrow\uparrow \quad E = -J \qquad \uparrow\downarrow \quad E = J.$

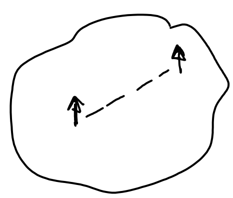
$$Z = \sum_C e^{-\beta H[C]} = \sum_{\sigma_1 \cdots\cdots \sigma_N} e^{-\beta H[\sigma_1 \cdots \sigma_N]}$$

$$F = -k_B T \operatorname{Log} Z.$$

$$\langle \sigma_{i_0} \rangle = \frac{\sum_{\sigma_1 - \sigma_N} \sigma_{i_0} \, e^{-\beta H[\sigma_1 - \sigma_N]}}{\sum_{\sigma_1 - \sigma_N} e^{-\beta H[\sigma_1 - \sigma_N]}}$$

$$= \frac{1}{Z} \sum_{\sigma_1 - \sigma_N} \sigma_{i_0} \, e^{-\beta H[\{\sigma\}]}$$

$\langle \sigma_{i_0} \sigma_{i_0 + r} \rangle$

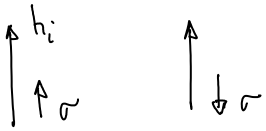


$$= \frac{1}{Z} \sum_{\{\sigma\}} \sigma_{i_0} \sigma_{i_0 + r} \, e^{-\beta H[\{\sigma\}]}$$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \sum_{\{\sigma\}} \mathcal{O}[\sigma] \, e^{-\beta H[\{\sigma\}]}$$

External field.

$$H = H_0[\{\sigma\}] - \sum_i h_i \sigma_i$$



$$Z[T, h_1, h_2 \text{-----} h_N] = \sum_{\{\sigma\}} e^{-\beta\left[H_0[\sigma] - \sum_i h_i \sigma_i\right]}$$

$$= [\;\;,\; 1, h_2 \cdots h_N] \qquad \{\sigma\}$$

$$\frac{\partial F[T, h_1 \cdots h_N]}{\partial h_{i_0}} \qquad F = -k_B T \log Z$$

$$= (-k_B T) \frac{1}{Z} \frac{\partial Z}{\partial h_{i_0}} = (-k_B T) \frac{1}{Z} \sum_{\{\sigma\}} \beta \sigma_{i_0} e^{-\beta [H_0 - \sum_i h_i \sigma_i]}$$

$$= \frac{-1}{Z} \sum_{\{\sigma\}} \sigma_{i_0} e^{-\beta [H_0[\sigma] - \sum_i h_i \sigma_i]}$$

$$\left. \frac{\partial F}{\partial h_{i_0}} \right|_{h_i = 0} = - \langle \sigma_{i_0} \rangle$$

$$\left. \frac{\partial F}{\partial h_{i_1} \partial h_{i_2}} \right|_{h=0} = \frac{\partial}{\partial h_{i_1}} \left[-\frac{1}{Z} \sum_{\{\sigma\}} \sigma_{i_2} e^{-\beta [H_0[\sigma] - \sum_i h_i \sigma_i]} \right]$$

$$= -\left\{ \frac{1}{Z} \sum_{\{\sigma\}} \sigma_{i_2} \sigma_{i_1} \beta e^{-\beta H} \right.$$

$$\left. + \left(\sum_{\{\sigma\}} \sigma_{i_2} e^{-\beta H} \right) \frac{-1}{Z^2} \left(\sum_{\{\sigma\}} \beta \sigma_{i_1} e^{-\beta H} \right) \right]$$

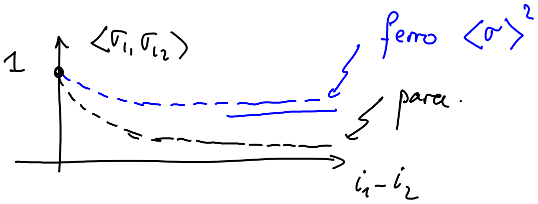
$$= -\beta \left[\langle \sigma_{i_1} \sigma_{i_2} \rangle - \langle \sigma_{i_1} \rangle \langle \sigma_{i_2} \rangle \right]$$

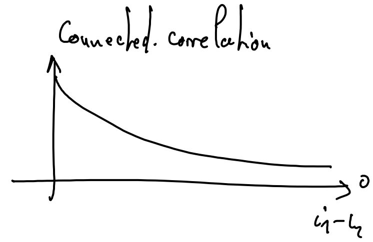
Connected correlation function.

$$i_1 - i_2 \to \infty \qquad \langle \sigma_{i_1} \sigma_{i_2} \rangle \longrightarrow \langle \sigma_{i_1} \rangle \langle \sigma_{i_2} \rangle$$

Ferromagnetic $\quad$ Paramagnetic $\quad T$

$\langle \sigma \rangle \neq 0 \quad T_c \quad \langle \sigma \rangle = 0$





$$\langle \sigma_{i_1} \sigma_{i_2} \rangle - \langle \sigma_{i_1} \rangle \langle \sigma_{i_2} \rangle = \left\langle \left(\sigma_{i_1} - \langle \sigma_{i_1} \rangle \right) \left(\sigma_{i_2} - \langle \sigma_{i_2} \rangle \right) \right\rangle$$

2) <u>External field ; Linear response.</u>

$H_0[\{\sigma\}]$

equi + external field $\longrightarrow$ Measure the response.

(magnetic, electric, ...)

$$H_0[\{\sigma\}] + \sum_i h_i O_i = H$$

$$\langle A \rangle_H$$

without the perturbation. $\langle A \rangle_{H_0} = 0$

$$\langle A_{i_0} \rangle_H = \langle A_{i_0} \rangle_{H_0} + a^d \sum_i \chi_{i_0 i} h_i + \cancel{h^2}$$

$$f(h_1, h_2, h_3) = \frac{\partial f}{\partial h_1}\bigg|_{h=0} h_1 + \frac{\partial f}{\partial h_2}\bigg|_{h=0} h_2 + \cdots \frac{\partial f}{\partial h_N}\bigg|_{h=0} h_N.$$

a : lattice spacing.

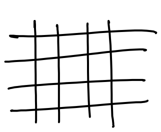
$$\langle A_{i_0} \rangle_H = a^d \sum_i \chi_{i_0 i} h_i + \cancel{\times}$$

if H_0 is invariant by translation $\chi_{i_0, i} \equiv \chi(i_0 - i)$

$$\langle A_{i_0} \rangle_H = a^d \sum_i \chi_{i_0 - i} h_i$$

$$\boxed{\begin{aligned} h_q &= a^d \sum_i e^{-iqr_i} h_i \\ h_i &= \frac{1}{\Omega} \sum_q e^{iqr_i} h_q. \end{aligned}}$$

$$\sum_i e^{iqr_i} = N \delta_{q,0}$$

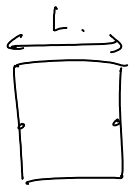


$a \to 0$ Continuum description.

$$a \sum_i \longrightarrow \int dx$$

$$h_q = \int d^d r\, e^{-iqr} h(r).$$

$$h(r) = \frac{1}{\Omega} \sum_q e^{iqr} h_q.$$



periodic boundary conditions.

$$r \to r + L \qquad h(r) = h(r+L)$$

$$\Rightarrow \quad e^{iqL} = 1 \qquad\qquad q_\alpha = \frac{2\pi}{L_\alpha} n_\alpha.$$

Continuum Limit:

$$\left| \begin{array}{lll} h_i \longrightarrow & h(r) & a \to 0 \\ h_q \longrightarrow & h(q). & L \to \infty. \end{array} \right.$$

$$\int d^d r \, e^{iqr} = \Omega \, \delta_{q,0}.$$

Relation:

$$\underset{q_x = \frac{2\pi}{L_x} n_x, \,----}{\frac{1}{\Omega} \sum_q} \longrightarrow \frac{1}{(2\pi)^d} \int d^d q.$$

$$\Omega \, \delta_{q,0} \longrightarrow (2\pi)^d \, \delta(q)$$

$$\langle A_{i_0} \rangle = \sum_i \chi_{i_0 - i} h_i$$

$$\langle A_q \rangle = a^d \sum_{i_0} e^{-iqr_{i_0}} \langle A_{i_0} \rangle$$

$$= a^d \sum_{i_0} e^{-iqr_{i_0}} \sum_i \chi_{i_0 - i} h_i$$

$$= a^d \sum_{i_0} e^{-iqr_{i_0}} \sum_i \chi_{i_0 - i} \frac{1}{\Omega} \sum_{q'} e^{iq'r_i} h_{q'}$$

$$= \frac{a^d}{\Omega} \sum_{\substack{i_0 \, i \\ q'}} e^{iq'r_i} e^{-iqr_{i_0}} \chi_{i_0 - i} h_{q'}.$$

$$r_{i_0} \longrightarrow r_{i_0} + r_i.$$

$$= \frac{a^d}{\Omega} \sum_{i_0 \, i} e^{iq'r_i} e^{-iq[r_{i_0} + r_i]} \chi_{i_0} h_{q'}$$

$$= a^d \sum_{i_0} e^{-iqr_{i_0}} \sum_i \chi_{i_0-i} \frac{1}{\Omega} \sum_{q'} e^{iq'r_i} h_{q'}$$

$$= \frac{a^d}{\Omega} \sum_{\substack{i_0\, i \\ q'}} e^{iq'r_i} e^{-iqr_{i_0}} \chi_{i_0-i} h_{q'}.$$

$$r_{i_0} \to r_{i_0} + r_i .$$

$$= \frac{a^d}{\Omega} \sum_{i_0\, i} e^{iq'r_i} e^{-iq[r_{i_0}+r_i]} \chi_{i_0} h_{q'}$$

$$= \frac{a^d}{\Omega} \sum_{i_0\, i} \underbrace{e^{i(q'-q)r_i}}_{\Omega \delta_{q,q'}} e^{-iqr_{i_0}} \chi_{i_0} h_{q'}$$

$$= a^d \sum_{i_0} e^{-iqr_{i_0}} \chi_{i_0}\, h_q = \chi_q h_q$$

$$\langle A_q \rangle = \chi_q h_q$$

Susceptibility of the system

$$h(x) = h_q \cos(qx)$$

$$A(x) = A_q \cos(qx)$$

$$A_q = \chi_q h_q$$

\# What are the susceptibilities?

$$\langle A_{i_0} \rangle \qquad\qquad H = H_0[\{\sigma\}] + \sum_i h_i O_i$$

$$\frac{1}{Z[h_i]} \left(\sum_{\{\sigma\}} A_{i_0} e^{-\beta[H_0 + \sum_i h_i O_i]} \right)$$

$$= \frac{\sum_{\{\sigma\}} A_{i_0} e^{-\beta H_0} \left(1 - \beta \sum_i h_i O_i\right)}{\sum_{\{\sigma\}} e^{-\beta H_0} \left(1 - \beta \sum_i h_i O_i\right)}$$

$$= \sum_{\{\sigma\}} A_{i_0} e^{-\beta H_0} - \beta \sum_i \sum_{\{\sigma\}} A_{i_0} h_i O_i \, e^{-\beta H_0}.$$

$$= \frac{\sum_{\{\sigma\}} A_{i_0}\, e^{-\beta H_0} - \beta \sum_i \sum_{\{\sigma\}} A_{i_0} h_i O_i\, e^{-\beta H_0}}{\underbrace{\sum_{\{\sigma\}} e^{-\beta H_0}}_{Z_0} - \beta \sum_i \sum_{\{\sigma\}} h_i O_i\, e^{-\beta H_0}}$$

$$= \frac{1}{Z_0} \sum_{\{\sigma\}} A_{i_0}\, e^{-\beta H_0} - \beta \sum_i h_i \sum_{\{\sigma\}} A_{i_0} O_i\, e^{-\beta H_0}$$

$$+ \sum_i \beta \frac{1}{Z_0^2} h_i \Big(\sum_{\{\sigma\}} A_{i_0}\, e^{-\beta H_0} \Big) \Big(\sum_{\{\sigma\}} O_i\, e^{-\beta H_0} \Big)$$

$$\langle A_{i_0} \rangle_H = \langle A_{i_0} \rangle_{H_0} - \beta \Big[\sum_i h_i \big(\langle A_{i_0} O_i \rangle - \langle A_{i_0} \rangle \langle O_i \rangle \big) \Big]$$

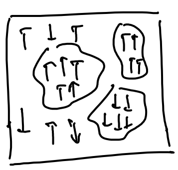
$$\langle A_{i_0} \rangle_H = \langle A_{i_0} \rangle_{H_0} - \sum_i h_i\, \chi_{i_0 i}$$

$$\chi_{i_0 i} = \beta \Big[\langle A_{i_0} O_i \rangle_{H_0} - \langle A_{i_0} \rangle_{H_0} \langle O_i \rangle_{H_0} \Big]$$

equilibrium.

Correlations at equilibrium.

$\beta \big[\langle A_{i_0} O_i \rangle - \langle A_{i_0} \rangle \langle O_i \rangle \big]$



response to a perturbation

apply h (couples to O_i)

$\downarrow$ measure response A_{i_0}

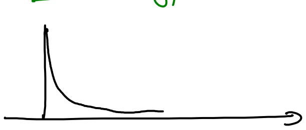
magnetic field $O_i = -\sigma_i$ measure. $m \to A_{i_0} = \sigma_{i_0}$.

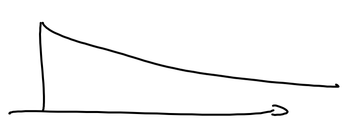
$$\beta \big[\langle \sigma_{i_0} \sigma_i \rangle - \langle \sigma_{i_0} \rangle \langle \sigma_i \rangle \big]$$

Fluctuation dissipation theorem.

Linear response $\Longleftrightarrow$ fluctuations at equilibrium

only hypothesis $\to$ Statistical equilibrium.





Small response.   Large response.

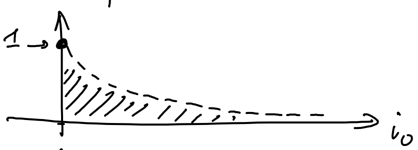
uniform magnetic field. $h_i \equiv h$.

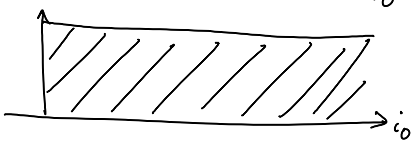
$\downarrow$ uniform magnetization response.

$$m_0 = \chi_{q=0}\, h_0. \qquad \chi_{q=0} = a^d \sum_{i_0} \chi_{i_0}\, e^{i \overset{0}{q}\, r_{i_0}}$$

$$= a^d \sum_{i_0} \chi_{i_0}.$$

Measures the integral of the function $\chi(i)$





$\langle \sigma_{i_0} \sigma_0 \rangle$

$i_0 = 0 \qquad \langle \sigma^2 \rangle = \langle 1 \rangle = 1$

$$\int_0^{+\infty} d^3 r \ \chi(r)$$

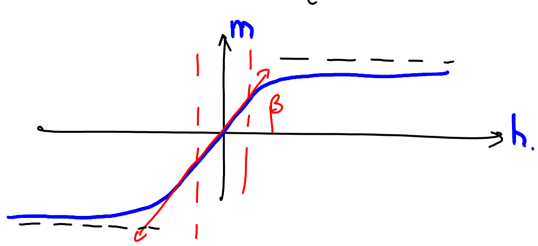
\# Example:

$$H_0 = 0 \qquad H = -h.\sigma$$

$$Z = \sum_{\{\sigma\}} e^{-\beta H} = \sum_{\{\sigma\}} e^{\beta h \sigma} = e^{\beta h} + e^{-\beta h}$$

$$\langle \sigma \rangle = \frac{\sum_{\{\sigma\}} \sigma\, e^{-\beta H}}{Z} = \frac{e^{\beta h} - e^{-\beta h}}{e^{\beta h} + e^{-\beta h}}$$

$$\langle \sigma \rangle = \text{th}(\beta h) \sim \beta h.$$

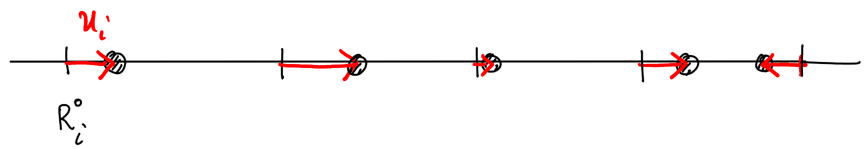


$$\chi = \beta \qquad \chi_{i_0 i} \equiv \beta \left[\langle \sigma_{i_0} \sigma_i \rangle - \langle \sigma_{i_0} \rangle \langle \sigma_i \rangle \right]$$

$$\chi^{\text{ED}} = \beta \left[\underset{1}{\langle \sigma^2 \rangle} - \underset{\overset{\|}{0}}{\langle \sigma \rangle^2} \right] \text{ without magnetic field } (h=0)$$

$$= \beta$$

3) Continuous systems; Functional integral



$\{u_1, u_2, \ldots\ldots, u_N\}$

$$H = \sum_i \frac{k}{2} (u_{i+1} - u_i)^2 . \qquad k: \text{spring constant.}$$

$$Z = \sum_{u_1, \ldots u_N} e^{-\beta H[u_1 \ldots u_N]}$$

$$x \gg a$$

$$u_{i+1} - u_i \approx a.(\nabla u) \qquad R_i^0 = a . i$$

$$H = \sum_i \frac{k}{2} a^2 (\nabla u_i)^2 . = \frac{1}{a} \int dr \, \frac{k}{2} a^2 (\nabla u(r))^2$$

$$= \frac{k}{2} \int dr \, \rho_0 (\nabla u(r))^2 \qquad \rho_0 : \text{density}$$

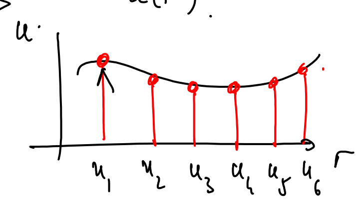
Define a partition function.

$$Z = \lim_{a \to 0} Z_a .$$

$$\langle O_i \rangle \longrightarrow \langle O(r_i) \rangle = \lim_{a \to 0} \langle O_i \rangle_a .$$

Configurations. :

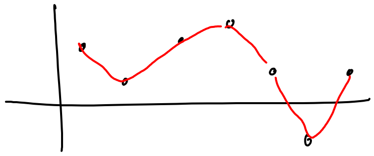
$$u_1, u_2 \ldots u_N \longrightarrow u(r) .$$



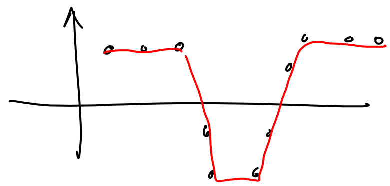
$$\sum_{u_1 u_2 u_3 u_4 .}$$

$$Z_a \equiv \int_{-\infty}^{+\infty} du_1 \int_{-\infty}^{+\infty} du_2 \int_{-\infty}^{+\infty} du_3 \ldots \int_{-\infty}^{+\infty} du_N \; e^{-\beta \frac{k}{2} \sum_i (u_{i+1} - u_i)^2}$$

$-\infty \qquad -\infty \qquad -\infty \qquad\qquad -\infty$



Conf 1



Conf 2

$$\int_{-\infty}^{+\infty} du_1 \int_{-\infty}^{+\infty} du_2 \cdots \int_{-\infty}^{+\infty} du_N \equiv \int \mathcal{D}u[r]$$

↳ Sum over all possible functions $u(r)$

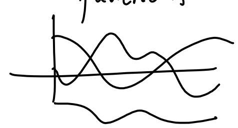
→ functional integral.

integral.

$$\underbrace{\int dx}_{\text{variable}} \underbrace{\frac{1}{1+x^2}}_{\text{function of } x}$$

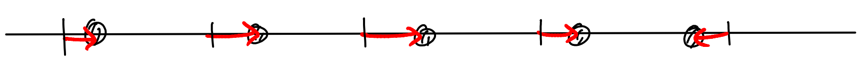
functional integral.

$$\underbrace{\int \mathcal{D}u(r)}_{\text{functions}} \; \underbrace{e^{-\beta H[u(r)]}}_{\text{functional of the function } u(r)}$$



Example:

$$H = \frac{1}{2} \int d^d r \int d^d r' \; V(r-r')\, u(r)\, u(r')$$



discrete system:

$$H = \frac{1}{2} a^{2d} \sum_{i,j} V_{i-j}\, u_i u_j$$

$$Z_a = \sum_{u_1, u_2 \ldots u_N} e^{-\beta \frac{a^{2d}}{2} \sum_{i,j} V_{i-j} u_i u_j}$$

Gaussian integrals

$$\frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} dx \sqrt{a}\; e^{-ax^2} = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} dy\; e^{-y^2} = \sqrt{\frac{\pi}{a}}$$

$$\int_{-\infty}^{+\infty} dx \quad x \quad e^{-x^2 a} \qquad = \qquad 0$$

$$\int_{-\infty}^{+\infty} dx \quad x^2 \quad e^{-\frac{1}{2}ax^2}$$

$$= \quad 0$$

$$\frac{\int_{-\infty}^{+\infty} dx \; x^2 \; e^{-\frac{1}{2}ax^2}}{\int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2 a} \; dx} = \frac{1}{a}.$$

$$e^{-\frac{1}{2}a\left[x+\frac{b}{a}\right]^2} e^{\frac{1}{2}\frac{b^2}{a}}$$

$$\frac{\int_{-\infty}^{+\infty} dx \; e^{-\frac{1}{2}ax^2 + bx}}{\int_{-\infty}^{+\infty} dx \; e^{-\frac{1}{2}ax^2}} = e^{\frac{b^2}{2a}}.$$

- <u>Multiple Gaussian Integral.</u>

$$\int \frac{du_1 \, du_2 \cdots du_N}{(2\pi)^{N/2}} \; e^{-\frac{1}{2}\sum_{i,j} u_i M_{ij} u_j}$$

λ_i eigenvalues of M.

$$\to \int \frac{d\alpha_1 \cdots d\alpha_N}{(2\pi)^{N/2}} \; e^{-\frac{1}{2}\sum_i \alpha_i \lambda_i \alpha_i}$$

$$= \prod_i \left(\int \frac{d\alpha_i}{(2\pi)^{1/2}} \; e^{-\frac{1}{2}\alpha_i \lambda_i \alpha_i} \right) \; \propto \prod_i \frac{1}{\sqrt{\lambda_i}}$$

$$\int \frac{du_1 - du_N}{(2\pi)^{N/2}} \; e^{-\frac{1}{2}\sum_{ij} u_i M_{ij} u_j + \sum_i u_i b_i} = \frac{1}{(\det M)^{1/2}} \; e^{\frac{1}{2}\sum_{ij} b_i M^{-1}_{ij} b_j}$$

$$\frac{\int \text{—} \; e^{-uMu + ub}}{\int \text{—} \; e^{-uMu}} = e^{\frac{1}{2}\sum_{ij} b_i M^{-1}_{ij} b_j}.$$

$$\frac{\int \text{—} \; u_{i_0} u_{i_1} \; e^{-\frac{1}{2}\sum_{ij} u_i M_{ij} u_j}}{\int \text{—} \; e^{-\frac{1}{2}\sum_{ij} u_i M_{ij} u_j}} = \left(M^{-1}\right)_{i_0 i_1}$$

$$Z_a = \sum_{u_1, u_2 \cdots u_N} e^{-\beta \frac{a^{2d}}{2} \sum_{ij} V_{i-j} u_i u_j}$$

$$a^{2d} \sum_{ij} V_{i-j} \, u_i u_j \longrightarrow \frac{1}{\Omega} \sum_q V_q \, u_q u_{-q}.$$

$$Z_a = \sum_{u_{q_1} u_{q_2} \cdots u_{q_N}} e^{-\beta \frac{1}{\Omega} \sum_q V_q u_q u_{-q}}$$

$$u_q = a^d \sum_j e^{-iqr_j} u_j$$

$$\sum_{u_1, u_2 \cdots u_N} = \int du_1 \int du_2 \cdots \int du_N .$$

$$\left[\int_{-\infty}^{+\infty} d\mathrm{Re}\, u_{q_1} \int_{-\infty}^{+\infty} d\mathrm{Im}\, u_{q_1} \cdots\cdots \int d\mathrm{Re}\, u_{q_N} \int d\mathrm{Im}\, u_{q_N} \right]$$

$$\overset{0}{|} \quad \overset{1}{|} \quad \cdots \quad \overset{N-1}{|} \qquad\qquad q = \frac{2\pi}{\underbrace{Na}_{L_x}} n$$

$$h(r_j) = \frac{1}{\Omega} \sum_q e^{iqr_j} h(q) \qquad r_j = a \cdot j$$

$$q \to q + \frac{2\pi}{a} \qquad e^{iq[a.j]} \to e^{iq[a.j]} e^{i\frac{2\pi}{a} a . j}$$

$$q \in \left[-\frac{\pi}{a}, \frac{\pi}{a} \right] \quad \text{Brillouin zone}$$

$$\Delta n = \frac{Na}{2\pi} \frac{2\pi}{a} = N$$

$$u_q^* = u_{-q}$$

On can limit to positive values of q. $\quad (u_q^* = u_{-q})$

N independent integrals

$$\prod_{q>0} \left(\int \frac{d\mathrm{Re}\, u_q \; d\mathrm{Im}\, u_q}{\pi} \right) \qquad \begin{cases} u = \mathrm{Re} + i\,\mathrm{Im} \\ u^* = \mathrm{Re} - i\,\mathrm{Im} . \end{cases}$$

$$y_1(x_1, x_2) \quad y_2(x_1, x_2) \qquad dx_1 dx_2 \to dy_1 dy_2 J$$

$$\begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} \end{vmatrix} \equiv J \qquad J = \begin{vmatrix} 1 & 1 \\ i & -i \end{vmatrix} = -2i$$

$$\iint \frac{d\mathrm{Re}\; d\mathrm{Im}}{\pi} \longleftrightarrow \iint \frac{du\, du^*}{2i\pi}$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{d\mathrm{Re}\; d\mathrm{Im}}{\pi} e^{-a u^* u} = \int \frac{d\mathrm{Re}\; d\mathrm{Im}}{\pi} e^{-a\left[\mathrm{Re}^2 + \mathrm{Im}^2\right]}$$

$$= \left(\int \frac{dR_e}{\sqrt{\pi}}\, e^{-aR_e^2}\right)\left(\int \frac{dI_m}{\sqrt{\pi}}\, e^{-aI_m^2}\right)$$

$$= \frac{1}{\sqrt{a}} \quad \frac{1}{\sqrt{a}}. = \frac{1}{a}.$$

$$\int \frac{du^* du}{2i\pi}. \qquad u = \rho e^{i\theta} \qquad u^* = \rho e^{-i\theta}.$$

$$\begin{vmatrix} e^{i\theta} & e^{-i\theta} \\ i\rho e^{i\theta}. & -i\rho e^{-i\theta} \end{vmatrix} \equiv -2i\rho.$$

$$\int \frac{du^* du}{2i\pi}. \rightarrow \int \frac{2i\rho}{2i\pi}\, d\rho\, d\theta = \int_0^{+\infty} \rho\, d\rho\, \frac{1}{\pi} \int_0^{2\pi} d\theta\, e^{-a\rho^2}$$

$$= 2 \int_0^{+\infty} \rho\, d\rho\, e^{-a\rho^2} = \int_0^{+\infty} dy\, e^{-ay} = \frac{1}{a}.$$

$$\boxed{\begin{gathered} \int \frac{du\, du^*}{2i\pi}\, e^{-a u^* u}. = \frac{1}{a}. \\[2ex] \frac{\int \frac{du\, du^*}{2i\pi} \begin{vmatrix} u^* \\ u \end{vmatrix} e^{-\frac{1}{2} a u^* u}}{\int \frac{du\, du^*}{2i\pi}\, e^{-\frac{1}{2} a u^* u}} = 0 \\[2ex] \frac{\int \frac{du\, du^*}{2i\pi}\, u u^*\, e^{-\frac{1}{2} a u^* u},}{\int \frac{du\, du^*}{2i\pi}\, e^{-\frac{1}{2} a u^* u}} = \frac{2}{a}. \end{gathered}}$$

$$\left(\prod_i \int \frac{du_i\, du_i^*}{2i\pi}\right) e^{-\sum_{ij} u_i^* M_{ij} u_j + \sum_i h_i^* u_i + h_i u_i^*} = \frac{1}{\det M}\, e^{\sum_{ij} h_i^* M_{ij}^{-1} h_j}$$

$$V_q = V_{-q}.$$

$$Z_a = \sum_{u_{q_1}\, u_{q_2} \cdots\cdots\, u_{q_N}} e^{-\beta \frac{1}{\Omega} \sum_q V_q\, u_q\, u_{-q}}$$

$$\prod_{q>0} \int \frac{dRe u_q\, dIm u_q}{\pi} \qquad e^{-\frac{\beta}{\Omega} \sum_q V_q\, u_q^* u_q}.$$

$$2\beta \quad \sum \quad V \quad u^* \; u$$

$$\prod_{q>0} \int \frac{d\mathrm{Re}\,u_q \, d\mathrm{Im}\,u_q}{\pi} \qquad e^{-\frac{\beta}{\Omega}\sum_q V_q \, u_q^* u_q}$$

$$\Big\| \quad \int \frac{du_q \, du_q^*}{2 i \pi} \qquad e^{-\frac{2\beta}{\Omega}\sum_{q>0} V_q \underbrace{u_q^* u_q}_{\mathrm{Re}\,u_q^2 + \mathrm{Im}\,u_q^2}}$$

$$= J \prod_{q>0} \left(\frac{\pi \Omega}{\beta V(q)} \right)$$

$$F = -k_B T \operatorname{Log} Z \qquad = -k_B T \sum_{q>0} \left[\operatorname{Log}\left(\frac{\pi\Omega}{\beta V(q)} \right) + \mathrm{Cst} \right]$$

$$= -k_B T \frac{\Omega}{\Omega} \sum_{q>0} \operatorname{Log}\left(\frac{1}{V(q)} \right) + \mathrm{Cst}' \Big]$$

$$= \Omega \left[-k_B T \, \frac{1}{\Omega} \sum_{q>0} \operatorname{Log}\left(\frac{1}{V(q)} \right) + \mathrm{Cst} \right]$$

$$= \Omega \left[-k_B T \int_{q>0}^{\pi/a} dq \, \operatorname{Log}\left(\frac{1}{V(q)} \right) + \mathrm{Cst} \right]$$

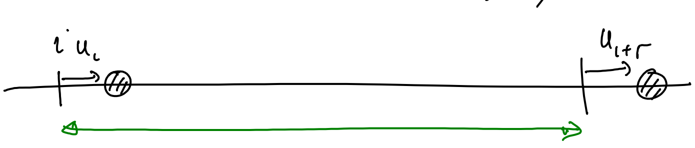
$$H = \sum_{ij} V(r_i - r_j) \, u_i \, u_j .$$

Correlation functions.

$$\langle u_{i_0} \rangle \qquad \langle u_{i_0}^2 \rangle \qquad \langle (u_{i_1} - u_{i_2})^2 \rangle$$

$$\langle u_{i_1} - u_{i_2} \rangle = \langle u_{i_1} \rangle - \langle u_{i_2} \rangle$$

$$B(r) = \langle (u_{i+r} - u_i)^2 \rangle$$



$$\langle \ldots \rangle = \frac{1}{Z} \sum_c e^{-\beta H} \; \ldots$$

$$B(r) = \frac{1}{Z} \sum_{u} e^{-\beta H[u]} \left(u_{j+r} - u_j\right)^2$$

$$H = \frac{1}{2} \sum_{ij} V_{i-j}\, u_i u_j \rightarrow \frac{1}{2}\frac{1}{\Omega} \sum_q V(q)\, u(q)\, u(-q)$$

$$u_j = \frac{1}{\Omega} \sum_q e^{iqr_j} u_q$$

$$\left(u_{j+r} - u_j\right) = \frac{1}{\Omega} \sum_q u_q \left(e^{iqr_{j+r}} - e^{iqr_j}\right)$$

$$B(r) = \frac{1}{\Omega^2} \sum_{q_1 q_2} \frac{1}{Z} \sum_u e^{-\beta H[u]} \left(e^{iq_1 r_{j+r}} - e^{iq_1 r_j}\right)$$
$$\left(e^{iq_2 r_{j+r}} - e^{iq_2 r_j}\right) u_{q_1} u_{q_2}$$

$$\langle u_{q_1} u_{q_2} \rangle = \langle u_{q_1} u^*_{-q_2} \rangle$$

$u(r)$ real

$u(q)$ $\quad u^*(q) = u(-q)$

$$H = \frac{1}{2\Omega} \sum_q V(q)\, u_q^* u_q = \frac{1}{2\Omega} \sum_{q>0} \left[V(q) + V(-q)\right] u_q^* u_q$$

$$= \frac{2}{2\Omega} \sum_{q>0} V(q)\, u_q^* u_q$$

$$V(r) = V(-r) \Rightarrow V(q) = V(-q).$$

$$\langle u_{q_1} u^*_{-q_2} \rangle = \frac{1}{Z} \underbrace{\int \mathcal{D}u[q]}_{\prod_{q>0} \int \frac{du_q\, du_q^*}{2i\pi}} e^{-\beta \frac{1}{\Omega} \sum_q V(q) u^*(q) u(q)} u_{q_1} u^*_{-q_2}$$

$$q_1 \neq -q_2 \quad : \frac{1}{Z}(\ldots)\left(\int du_{q_1} du^*_{q_1}\, e^{\cdots}\, u_{q_1}\right)\left(\int du_{q_2} du^*_{q_2}\, e^{\cdots}\, u^*_{-q_2}\right)$$

$$q_1 \neq -q_2 \Rightarrow \langle u_{q_1} u^*_{-q_2} \rangle = 0$$

$$(q_1 = -q_2) \Rightarrow \quad \frac{1}{Z}\left(\int \frac{da_q\, da_q^*}{2i\pi}\, e^{-\frac{2\beta}{2\Omega} V(q) u_q^* u_q}\right)_{q \neq q_1} \left(\int \frac{da_{q_1} da_{q_1}^*}{2i\pi}\, e^{-\frac{2\beta}{2\Omega} V(q) u_{q_1}^* u_{q_1}}\, u_{q_1} u_{-q_1}^*\right)$$

$$Z = \prod_{q>0} \left(\int \frac{da_q\, da_q^*}{2i\pi}\, e^{-\frac{2\beta}{2\Omega} V(q) u_q u_q^*}\right)$$

$$e^{-\frac{2\beta}{2\Omega}\sum_{q>0} V(q) u_q^* u_q} = \prod_{q>0} e^{-\frac{2\beta}{2\Omega} V(q) u_q^* u_q}.$$

$$\langle u_{q_1} u_{-q_1}^* \rangle = \frac{\int (--)\, e^{-\frac{2\beta}{\Omega} V(q_1) u_{q_1}^* u_{q_1}}\, u_{q_1} u_{q_1}^*}{\int \frac{da_{q_1} da_{q_1}^*}{2i\pi}\, e^{-\frac{2\beta}{\Omega} V(q_1) u_{q_1}^* u_{q_1}}} = \frac{\Omega}{\beta V(q)}.$$

$$\boxed{\langle u(q_1) u^*(q_2) \rangle = \frac{\int \mathcal{D}u[q]\; u(q_1) u^*(q_2)\; e^{-\frac{1}{2}\sum_q A(q) u^*(q) u(q)}}{\int \mathcal{D}u[q]\; e^{-\frac{1}{2}\sum_q A(q) u^*(q) u(q)}} = \frac{1}{A(q_1)}\,\delta_{q_1, q_2}}$$

$$B(r) = \frac{1}{\Omega^2}\sum_{q_1 q_2} \frac{1}{Z}\sum_u e^{-\beta H[u]} \left(e^{iq_1 r_{j+r}} - e^{iq_1 r_j}\right)\left(e^{iq_2 r_{j+r}} - e^{iq_2 r_j}\right) u_{q_1} u_{q_2}$$

$$= \frac{1}{\Omega^2}\sum_{q_1 q_2} \left(e^{iq_1 r_{j+r}} - e^{iq_1 r_j}\right)\left(e^{iq_2 r_{j+r}} - e^{iq_2 r_j}\right) \langle u_{q_1} u_{q_2} \rangle$$

$$\langle u_{q_1} u_{-q_2}^* \rangle = \delta_{q_1, -q_2} \frac{\Omega}{\beta V(q)}$$

$$= \frac{1}{\Omega}\sum_{q_1} \left(e^{iq_1 r_{j+r}} - e^{iq_1 r_j}\right)\left(e^{-iq_1 r_{j+r}} - e^{-iq_1 r_j}\right) \frac{1}{\beta V(q_1)}$$

$$= \frac{1}{\Omega}\sum_{q_1} \left[2 - e^{iq_1 (r_{j+r} - r_j)} - e^{iq_1 (r_j - r_{j+r})}\right] \frac{1}{\beta V(q_1)}$$

$$= \frac{1}{\,}\sum \left[2 - 2\cos(q_1 (r_{j+r} - r_j))\right] \frac{1}{\,}$$

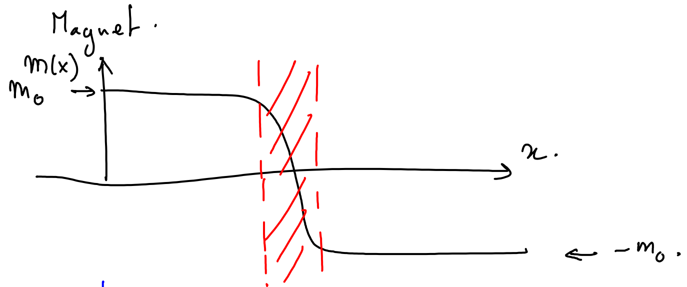
$$= \frac{1}{\Omega}\sum_{q_1}\left[2 - 2\cos\left(q_1\big(\underbrace{r_{j+r}-r_{j'}}_{r}\big)\right)\right]\frac{1}{\beta V(q_1)}$$

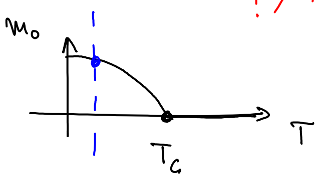
$$B(r) = \frac{1}{\Omega}\sum_{q}\left[1-\cos(qr)\right]\frac{2}{\beta V(q)}$$

$$\to \frac{1}{2\pi}\int dq\,\left[1-\cos(qr)\right]\frac{2}{\beta V(q)}$$

4) Example: (Disordered) Elastic Systems

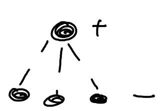
Magnet.





$T \ll T_c$

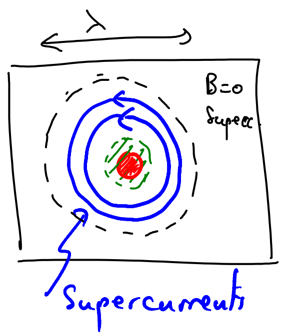
Ferroelectric



$\uparrow \vec{P}$ $\vec{P}$: electric dipole moment

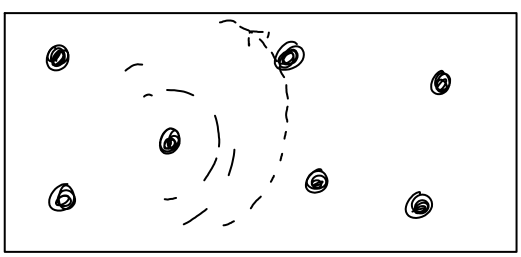
Vortices

λ: penetration length.

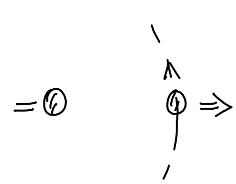


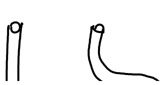
● Normal.
Size ξ_0
(cooper pair)

(Screen B)

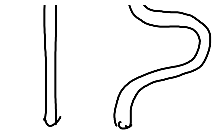


Abrikosov Vortices

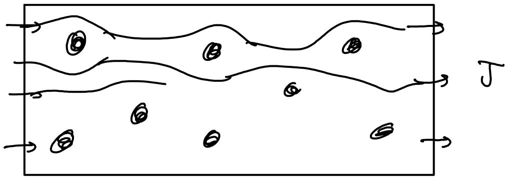


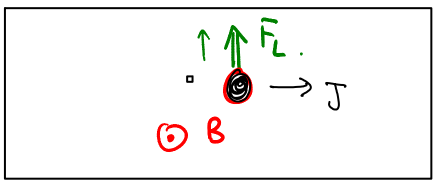


$E \propto$ Length.



E_{min} $\quad E > E_{min}$.





$F_L = \vec{J} \wedge \vec{B}$

$E = V . B$

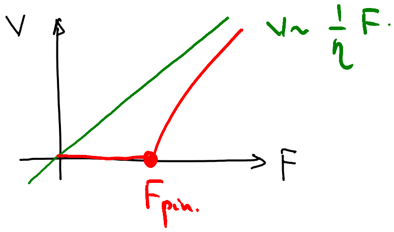
$J \longrightarrow E$

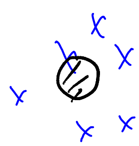
$V \propto F_L$

$\rho = \rho_N \left(\frac{H}{H_{c2}} \right)$ Bardeen Stephen. Resistance.

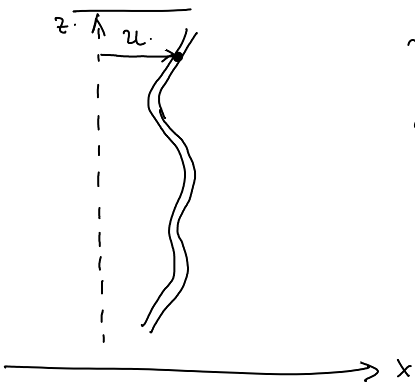
Pin the vortices $\quad V \simeq 0 \quad E \simeq 0$

↳ Disorder the superconductor !!



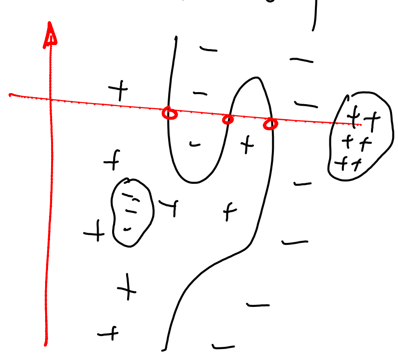


\# Model ?



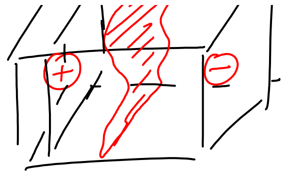
$u(z)$

domain wall cost ε_0 per unit length



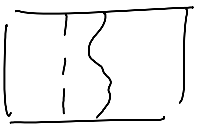
{ No bubbles
{ No overhangs.

Space of d dimensions.

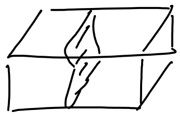


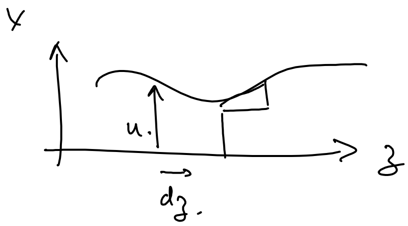
"plane" of $d-1$ dimensions
$(y, z, \ldots)$ $\quad u(y, z \ldots)$.

$d=2$



$d=3$





$$l^2 = dz^2 + du^2$$

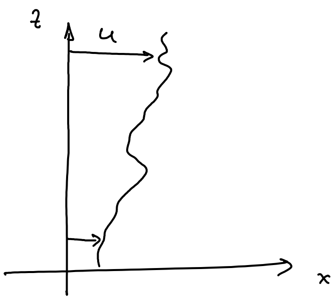
$$l = \sqrt{dz^2 + du^2} = dz\sqrt{1 + \left(\frac{du}{dz}\right)^2}$$

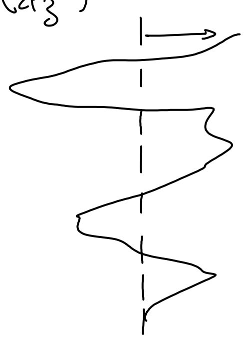
$$\text{Length} = \int_{-\infty}^{+\infty} dz \sqrt{1 + \left(\frac{du}{dz}\right)^2}$$

$$\Delta E = E(u \neq 0) - E(u=0) = \varepsilon_0 \int_{-\infty}^{+\infty} dz \left[\sqrt{1 + \left(\frac{du}{dz}\right)^2} - 1\right]$$

u small $\quad$ elastic approximation

$$= \frac{\varepsilon_0}{2} \int_{-\infty}^{+\infty} dz \left(\frac{du}{dz}\right)^2$$





$$B(z) = \langle (u(z) - u(z=0))^2 \rangle$$

roughness of the domain wall.

$$\begin{cases} H = \frac{c}{2} \int_{-\infty}^{+\infty} dz \, (\nabla_z u)^2 \\ B(z) = \langle [u(z) - u(0)]^2 \rangle = ? \end{cases}$$

$$u(z) = \frac{1}{\Omega} \sum_q e^{iqz} u_q. \qquad \nabla_z u(z) = \frac{1}{\Omega} \sum_q e^{iqz} (iq) u_q.$$

$$H = \frac{c}{2} \int d^d z \, \frac{1}{\Omega^2} \sum_{q_1 q_2} e^{iq_1 z} e^{iq_2 z} \, iq_1 \, iq_2 \, u_{q_1} u_{q_2}$$

$$H = \frac{c}{2} \int d^d_z \, \frac{1}{\Omega^2} \sum_{q_1 q_2} e^{i q_1 z} e^{i q_2 z} \, i q_1 \, i q_2 \, u_{q_1} u_{q_2}$$

$$\int d_z \, e^{i(q_1 + q_2) z} \equiv \Omega \, \delta_{q_1 + q_2}$$

$$H = \frac{c}{2} \frac{1}{\Omega} \sum_{q_1} q_1^2 \, u_{q_1} u_{q_1} \equiv \frac{c}{2\Omega} \sum_q q^2 u_q^* u_q.$$

$d = 2, 3 \ldots \quad q_x^2 + q_y^2 + \ldots$

$$B(r) = \frac{1}{\Omega} \sum_q \left[1 - \cos(q r) \right] \frac{2}{\beta V(q)}$$

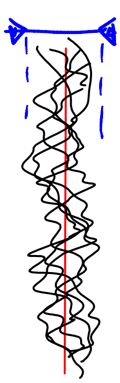
$$B(z) = \frac{1}{\Omega} \sum_q \left[1 - \cos(q z) \right] \frac{2}{\beta q^2 c}$$

$$B(z) = \frac{T}{c} \frac{2}{(2\pi)^d} \int_0^\Lambda d^d q \, \left(1 - \cos(q \cdot z) \right) \frac{1}{q^2}.$$

Λ is coming from the finite width of the line.

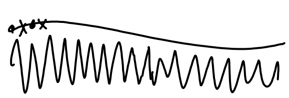
$$\begin{aligned} B(z) &= \langle u(z)^2 + u(0)^2 - 2u(z) u(0) \rangle \\ &= \langle u^2(z) \rangle + \langle u(0)^2 \rangle - 2 \langle u(z) u(0) \rangle \\ &= 2 \langle u(0)^2 \rangle - 2 \langle u(z) u(0) \rangle \end{aligned}$$

$$\begin{aligned} B(z \to \infty) &= 2 \langle u(0)^2 \rangle - 2 \langle u(z) \rangle \langle u(0) \rangle \\ &\to 2 \langle u(0)^2 \rangle \end{aligned}$$



$\langle u(0)^2 \rangle$

$$\int dq \, f(q) \cos(q z)$$

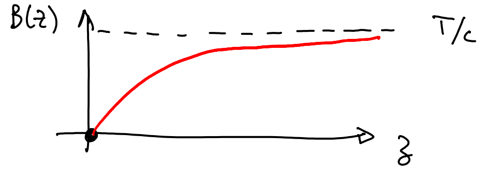


$$B(z \to \infty) = \frac{2T}{(2\pi)^d c} \int_0^\Lambda d^d q \, \frac{1}{q^2}$$

- $d = 3 \quad \langle u^2 \rangle \to$ finite $\propto T/c$

the wall remains "flat"

l effective width $\propto \frac{T}{c}$, thermal fluctuations.



- $\underline{d=2}$: $\langle u^2 \rangle = \infty$!! $\longrightarrow$ Wall becomes rough.

$$B(z) = \frac{T}{c} \frac{2}{(2\pi)^2} \int^{\Lambda} d^2q \, \frac{1}{q^2} \left(1 - \cos(\vec{q}\vec{z})\right)$$

$qz \ll 1 \qquad 1 - \cos(qz) \approx \frac{q^2 z^2}{2}$.

$qz \gg 1 \qquad 1 - \cos(qz) \approx 1$

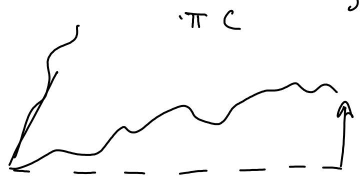
$$B(z) \propto \frac{2}{(2\pi)^2} \frac{T}{c} \left[\int_{qz \ll 1} d^2q \, \frac{z^2}{2} + \int_{qz \gg 1}^{\Lambda} \frac{d^2q}{q^2} \right]$$

$$\underbrace{(1/z)^2 \cdot \left(\frac{z^2}{2}\right)}_{qz \sim 1} + \int_{q \sim 1/z}^{\Lambda} \frac{d^2q}{q^2}$$

$$(2\pi) \int_{1/z}^{\Lambda} \frac{dq}{q} \simeq (2\pi) \operatorname{Log}\left[\frac{\Lambda}{1/z}\right]$$

$$B(z) = \frac{2T}{(2\pi)c} \left[\operatorname{Log}(\Lambda z) + C \right]$$

$$\simeq \frac{T}{\pi c} \operatorname{Log}[\Lambda' z]$$



- $\underline{d=1}$: $\langle u^2 \rangle = \infty$

$$B(z) = \frac{T}{c} \frac{2}{(2\pi)^2} \int^{\Lambda} d^2q \, \frac{1}{q^2} \left(1 - \cos(\vec{q}\vec{z})\right)$$

$$B(z) = \frac{T}{c}\,\frac{2}{(2\pi)^2}\left[\int_0^{1/z}\frac{dq}{q^2}\,\frac{q^2 z^2}{2} + \int_{1/z}^{\Lambda} dq\,\frac{1}{q^2}\right]$$

$$= \frac{T}{c}\,\frac{2}{(2\pi)^2}\left[\cdot\,\frac{z}{2} + \left(\frac{-1}{q}\right)\Big|_{1/z}^{\Lambda}\right]$$

$$= \frac{T}{c}\,\frac{2}{(2\pi)^2}\left[\cdot\, z + z - \frac{1}{\Lambda}\right]$$

$$\propto \text{Cst}\,\frac{T}{c}\,z \qquad z>0.$$

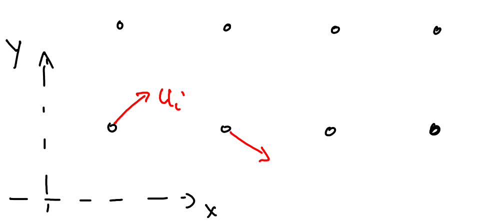
$B(z) \propto z$ Define a roughness exponent ζ

$B(z) \sim z^{2\zeta}$ $\quad B(z) \sim \langle (u(z)-u(0))^2 \rangle$

$u(z) \sim z^{\zeta}$

$\zeta = 1/2$ Roughness due to thermal fluctuations

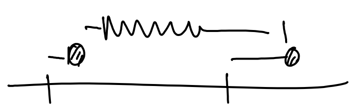
Crystals:



$R_i + u_i$

$\uparrow$ equ.

$u_i \begin{cases} u_x \\ u_y \\ u_z \end{cases}$

Energy. 

$$(u_{i+1} - u_i)^2 \longrightarrow (\nabla u)^2.$$

$$E[u_1 \dots u_N] + \mathcal{O}(u) + \mathcal{O}(u\,u)$$

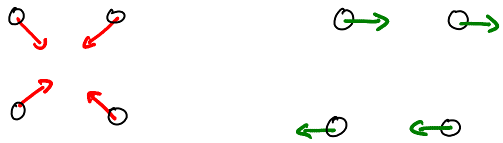
$\uparrow_{u=0}$ min. $\quad \mathcal{O}(u) = 0$

Energy is a quadratic function of $(\nabla . u)$ (elastic approx).

$\nabla_\alpha u_\beta$ $\qquad \alpha\beta\gamma\delta \,\big|\, x, y, z$

$$E = \frac{1}{2}\sum C_{\alpha\beta\gamma\delta}\,(\nabla^\alpha u^\beta)(\nabla^\gamma u^\delta)$$

$$E = \frac{1}{2} \sum_{\substack{\alpha\beta \\ \gamma\delta}} C_{\alpha\beta\gamma\delta} \, (\nabla u)(\nabla u)$$



Compression.

$\delta\rho \neq 0$

$\mathrm{div}\, u \neq 0$

Shear. $(\nabla_y u_x) \neq 0$

$\delta\rho = 0$

For simplicity : u as a scalar

$$E = \frac{\rho}{2} \int \left[(\nabla_x u)^2 + (\nabla_y u)^2 + (\nabla_z u)^2 \right] d^d r$$

$u(x=i, y)$ $\begin{vmatrix} u_x \\ 0 \\ 0 \end{vmatrix}$

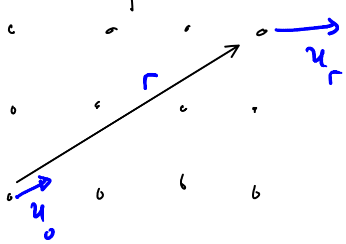
Continuum limit

$\downarrow$

$u(x, y)$

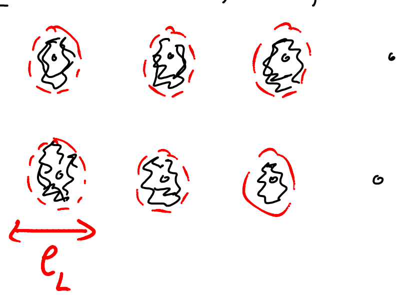
$$E = \frac{c}{2} \int d^d r \left[(\nabla_x u)^2 + (\nabla_y u)^2 + \dots \right]$$

$$= \frac{c}{2\Omega} \sum_q (q_x^2 + q_y^2 + \dots) \, u_q^* u_q.$$

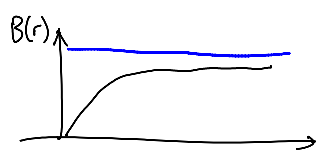


$$B(r) = \left\langle \left[u(r) - u(0) \right]^2 \right\rangle$$

$\underline{d=3}$ $B(r=\infty)$ is finite $\langle u^2(0) \rangle \equiv \frac{T}{c}$ Cste



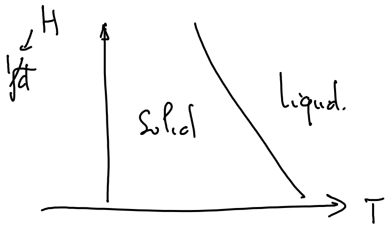
perfectly ordered.



$$\ell_L = \sqrt{\langle u^2 \rangle} \Rightarrow \ell_L^2 = \frac{T}{c} \text{ cste}$$

Lindemann length.

Crystal will melt when $\ell_L \sim c_L \, a$
(Lindemann Criterion) $\qquad c_L \simeq 0,1$



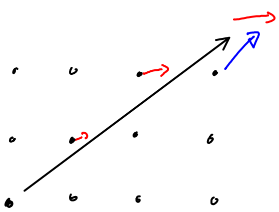
c practically independent of ρ

$H \uparrow \quad \rho \uparrow \quad a \searrow \quad$ melts

$\underline{d=3}$: perfect positional order in a crystal

$\underline{d=2}$ $\qquad \langle u^2 \rangle = \infty$!

$$B(z) \sim \frac{T}{c} \, \mathrm{Log}[\Lambda z]$$



⚠ $\nabla u \ll 1$.

$$u(z)^2 \sim \mathrm{Ln}[z] \Rightarrow u(z) \sim \mathrm{Log}^{1/2}[z]$$

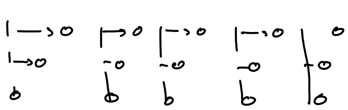
$$\nabla u(z) \sim \frac{1}{\mathrm{Ln}(z)^{1/2}} \, \frac{1}{z}$$

In $d \leq 2$ thermal fluctuations always destroy the perfect positional order !!

[General Situation: | Mermin-Wagner "theorem".
| Goldstone modes.

| Continuous Symmetry $\longrightarrow$ Spontaneous breaking of Symmetry.

[Transl Sym $\longrightarrow$ Crystal.]

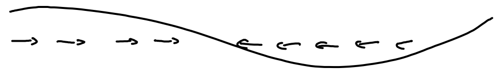


Global $u \Rightarrow$ Same energy

existence of the Sym $\Rightarrow$ Modes of excitations that cost little energy when they are of long wavelength.

$$\mathcal{E}(q) \to 0 \qquad q \to 0.$$

$$E = \frac{1}{\Omega} \sum_q q^2 u_q^* u_q .$$



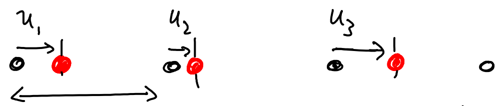
$$T \int \frac{d^d q}{\varepsilon(q)}$$

$d \leq 2 \rightarrow$ divergences || Restoration of the symmetry

$d > 2 \rightarrow$ OK.

Density in a Crystal.

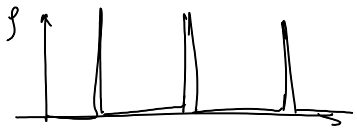
$$\rho(r) = \sum_i \delta(r - R_i - u_i)$$



δu small at scale a.

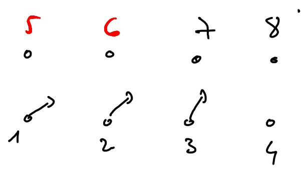
X ray, Neutrons, etc $\rightarrow \rho(q) \rightarrow \langle |\rho(q)|^2 \rangle$

$u_i \longrightarrow u(r)$ Continuous limit



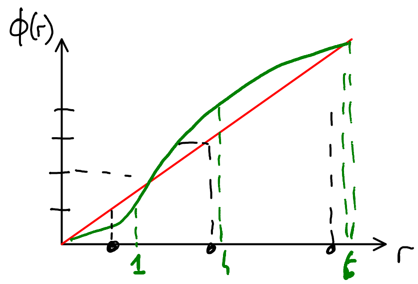
$$\sum_i (u_{i+1} - u_i)^2 \qquad u_{i+1} - u_i \ll 1$$

(∇u) is nearly constant at the scale a (lattice spacing)



$\phi(r)$. | $2\pi n$. when. r is the position of the nth particle

$$\phi(r) = r - u(\phi(r))$$



$$\rho(r) = \sum_i \delta(r - R_i - u_i)$$

$$\sum_n |\nabla\phi(r)| \, \delta(\phi(r) - n)$$

$$\delta(f(x)) = \sum_{\text{zero } f} \frac{1}{|f'(x_i)|} \delta(x - x_i)$$

$$\rho(r) = \sum_n |\nabla\phi(r)| \, \delta(\phi(r) - n)$$

$$= |\nabla\phi(r)| \sum_n \delta(\phi(r) - n)$$

$$= |\nabla\phi(r)| \bullet \sum e^{ip\phi(r)}$$

$$= |\nabla\phi(r)| \cdot \sum_p e^{ip\,\phi(r)}$$

$$\phi(r) \equiv \rho_0 r - \delta\phi(r).$$

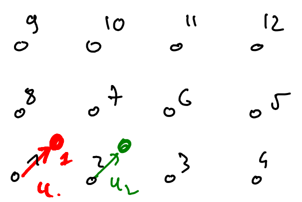
$$\rho(r) = \left[\rho_0 - \nabla\delta\phi(r)\right] \sum_p e^{ip\left[\rho_0 r - \underbrace{\delta\phi(r)}\right]}$$

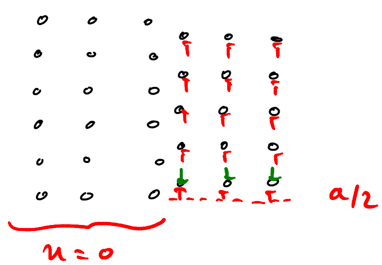
$$\delta\phi(r) \equiv \rho_0 u(r).$$

$$\rho(x) = \underset{\uparrow}{\rho_0} - \rho_0 \nabla u(r) + \rho_0 \sum_K e^{iK\left[r - u(r)\right]}$$

average density

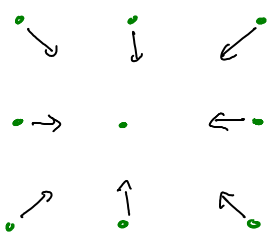
vectors of the reciprocal lattice





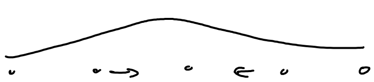
Number the particles $\Leftrightarrow$ displacements are uniquely defined

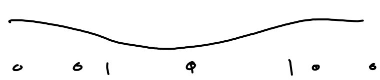
$\Leftrightarrow$ perfect topological order
(no dislocations, disclinations)



$\rho(x=0) \nearrow \quad \nabla \cdot u$

density seen at lengthscales $\gg a$.

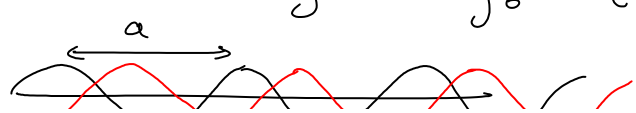


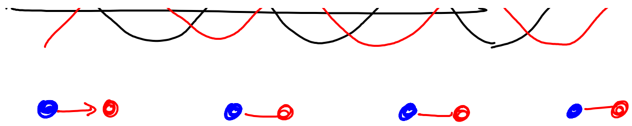


$$\rho_0 \sum_K e^{iK[r - u(r)]}$$

$$K = \frac{2\pi}{a}. \qquad \longrightarrow \qquad \rho(x) = \rho_0 \cos\left(\frac{2\pi}{a}\left(r - u(r)\right)\right)$$

$u = 0$





$u \approx$ cste at scale x.

$$H = \frac{c}{2} \int d^d r \, (\nabla u)^2$$

$$\rho(q) = \int d^d r \, e^{iqr} \rho(r) \qquad \langle |\rho_q|^2 \rangle$$

$$\rho(r) = \rho_0 - \rho_0 \nabla u + \rho_0 \sum_K e^{iK[r - u(r)]}$$

$$\int dr \, e^{iqr} \nabla u = - \int dr \, \nabla e^{iqr} u = - i \vec{q} \cdot \vec{u}_q.$$

$$\langle |\rho_q|^2 \rangle \equiv q^2 \langle u_q^* u_q \rangle = q^2 \frac{1}{Z} \int \mathcal{D}u \, e^{-\frac{c}{2T} \int dr (\nabla u)^2} u_q^* u_q$$

$$\frac{1}{Z} \int \mathcal{D}u_q \mathcal{D}u_q^* \, e^{-\frac{c}{2T\Omega} \sum_{q'} q'^2 u_{q'}^* u_{q'}}. \qquad u_q^* u_q = \frac{T\Omega}{c q^2}$$

$$\langle |\rho_q|^2 \rangle \equiv \Omega \frac{T}{c}$$

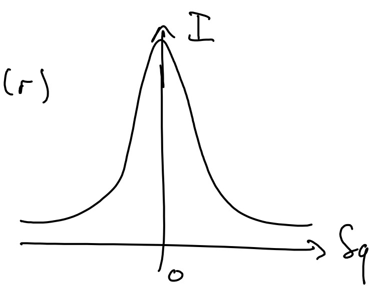
$$\rho_q = \int dr \, e^{iqr} \sum_K e^{iK(r - u(r))}$$

$$= \sum_K \int dr \, e^{i(q-K)r} e^{iKu(r)}.$$

$\underline{u=0}$ $$\rho_q = (2\pi)^d \sum_K \delta(q - K)$$

$$q = K + \delta q.$$

$$\rho_{K_0 + \delta q} = \int dr \, e^{i \delta q \, r} e^{iK_0 u(r)}$$



isolate one peak.

$$\langle |\rho_{K_0 + \delta q}|^2 \rangle = \int dr_1 \int dr_2 \, e^{i \delta q (r_1 - r_2)} \langle e^{iK_0 u(r_1)} e^{-iK_0 u(r_2)} \rangle$$

$$\int dr_1 \int dr_2 \, e^{i \delta q (r_1 - r_2)} \langle e^{iK_0 (r_2 + r_1 - r_2)} e^{-iK_0 u(r_2)} \rangle$$

$$\Omega \int dr \; e^{i\delta q r} \; \left\langle e^{iK_0 u(r)} \; e^{-iK_0 u(0)} \right\rangle$$

$$\left\langle e^{iK_0[u(r)-u(0)]} \right\rangle = \frac{1}{Z}\int \mathcal{D}u \; e^{-\frac{c}{2T}\int dr' (\nabla u)^2_{r'}} \; e^{iK_0[u(r)-u(0)]}$$

$$\frac{1}{Z}\int \mathcal{D}u_q^* \mathcal{D}u_q \; e^{-\frac{c}{2T\Omega}\sum_q q^2 u_q^* u_q} \; e^{iK_0[u(r)-u(0)]}$$

$$\frac{1}{Z}\int \mathcal{D}u_q^* \mathcal{D}u_q \; e^{-\frac{c}{2T\Omega}\sum_q q^2 u_q^* u_q + \frac{iK_0}{\Omega}\sum_q (e^{iqr}-1)u_q}$$

$$\frac{1}{Z}\int \mathcal{D}u_q^* \mathcal{D}u_q \; e^{-\frac{c}{2T\Omega}\sum_q q^2 u_q^* u_q + \frac{iK_0}{2\Omega}\sum_q (e^{iqr}-1)u_q + \frac{iK_0}{2\Omega}\sum (e^{-iqr}-1)u_q^*}$$

$$\frac{1}{Z}\int \mathcal{D}u_q^* \mathcal{D}u_q \; e^{\left\{-\frac{c}{2T\Omega}\sum_q q^2\left[u_q^* + \frac{iK_0T}{q^2c}(e^{iqr}-1)\right]\left[u_q + \frac{iK_0T}{q^2c}(e^{-iqr}-1)\right] - \frac{c}{2T\Omega}\sum_q q^2 \frac{K_0^2T^2}{q^4c^2}(e^{iqr}-1)(e^{-iqr}-1)\right\}}$$

$$= \frac{1}{Z}\int \mathcal{D}\tilde{u}_q^* \mathcal{D}\tilde{u}_q \; \exp\left[-\frac{c}{2T\Omega}\sum_q q^2 \tilde{u}_q^* \tilde{u}_q - \frac{K_0^2 T}{2c\Omega}\sum_q \left(2-2\cos(qr)\right)\frac{1}{q^2}\right]$$

$$= \; e^{-\frac{K_0^2T}{2c\Omega}\sum_q (2-2\cos(qr))\frac{1}{q^2}}$$

$$e^{-\frac{1}{2}K_0^2 B(r)}.$$

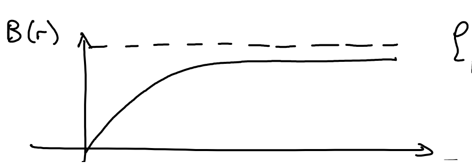
$$\left\langle e^{iK_0 u(r)} \; e^{-iK_0 u(0)} \right\rangle = e^{-\frac{1}{2}K_0^2 \left\langle [u(r)-u(0)]^2 \right\rangle}$$

$\rightarrow$ quadratic H in u $\qquad$ exp [linear form of u]

$\qquad\qquad\qquad\qquad$ = exp [$\frac{1}{2}$ $\langle$ Square $\rangle$]

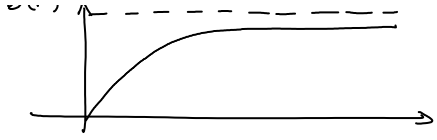
$$\left\langle |\rho_{K_0+\delta q}|^2 \right\rangle \frac{1}{\Omega} \equiv \int dr \; e^{i\delta q r} \; e^{-\frac{1}{2}K_0^2 B(r)}$$

<u>$d=3$</u>



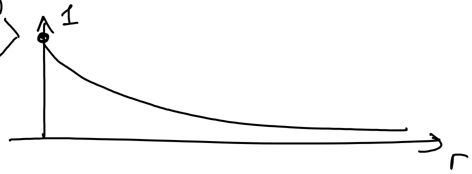
$\xi_L^2 \propto \frac{T}{c}$ Cst

$d=3$

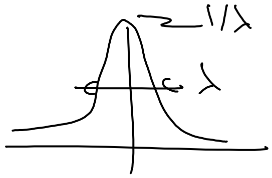


$\xi_L \propto \frac{1}{c}$ cst

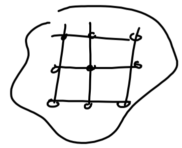
$\langle e^{iK(u(r))} e^{-iK u(0)} \rangle = C(r)$



$$\langle |\;|^2 \rangle = \frac{\lambda^{-1}}{\delta q^2 + \lambda^{-2}}$$



$\hookrightarrow \quad C(r) \sim e^{-r/\lambda}$



$d=3$

$\simeq e^{-\frac{1}{2}K_0^2 \rho_L^2}$

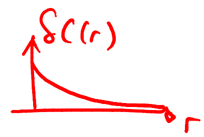
$\rho_L^2 \sim \frac{T}{c}$

$$\int_{-\infty}^{+\infty} dr\, e^{-|r|/\lambda} e^{iqr} = \int_0^{+\infty} dr\, e^{-r/\lambda} e^{iqr} + \int_{-\infty}^{0} dr\, e^{-|r|/\lambda} e^{iqr}$$

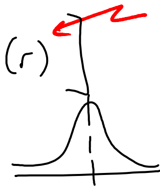
$$= \frac{-1}{iq - 1/\lambda} - \int_{+\infty}^{0} dr\, e^{-r/\lambda} e^{-iqr}$$

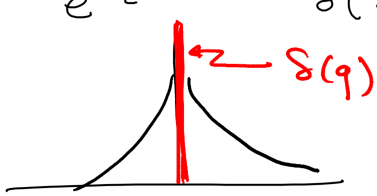
$$= \frac{1}{1/\lambda - iq} + \frac{1}{1/\lambda + iq} = \frac{2/\lambda}{(1/\lambda)^2 + q^2}$$

$$\langle |\rho_q|^2 \rangle \propto \int dr\, e^{i\delta q r} \left[e^{-\frac{1}{2}K_0^2 \rho_L^2} + \delta C(r) \right.$$



$$= e^{-\frac{1}{2}K_0^2 \rho_L^2}\, \delta(\delta q) +$$



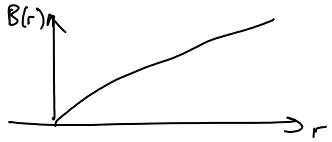


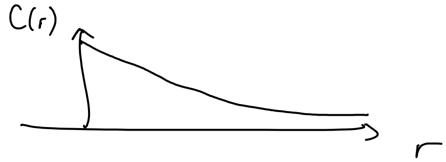
$d=3$: finite temperature $\Rightarrow$ No broadening of the peak
reduction of the weight (Debye-Waller factor)

$d=2$: $\quad B(r) = \frac{T}{c} \operatorname{Log}[r/a]$

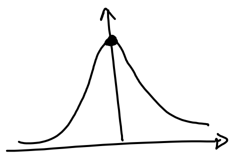
$$\langle |\rho_{K_0+\delta q}|^2 \rangle \equiv \int dr \; e^{i\delta q r} \; e^{-\frac{1}{2}K_0^2 \frac{T}{C} \log(r/a)}$$

$$= \int_{r>a} d^2r \; e^{i\delta q r} \left(\frac{a}{r}\right)^{\frac{K_0^2 T}{2C}}.$$

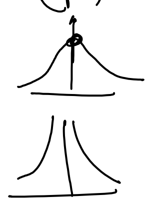




$\underline{\delta q = 0}$



$$\int d^2r \left(\frac{a}{r}\right)^{\nu} \qquad \nu = \frac{K_0^2 T}{2C}$$



$\nu > 2$ integral converges

$\nu < 2$ integral diverges

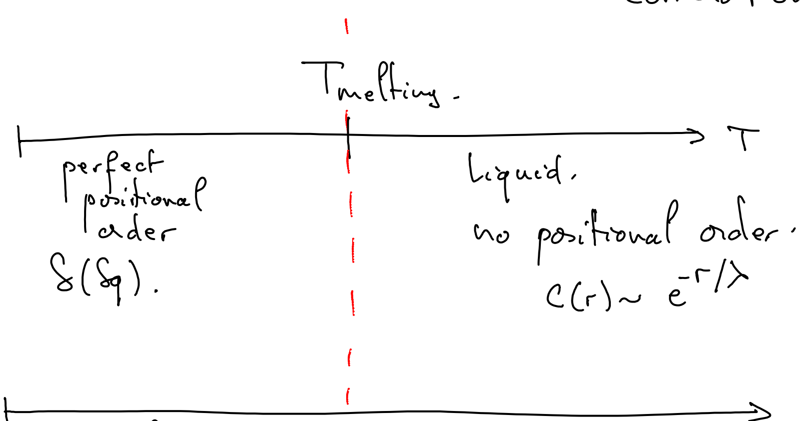
$$\int d^2r \left(\frac{1}{r}\right)^{\nu} e^{i\delta q r}. \approx \int_a^{1/\delta q} d^2r \left(\frac{1}{r}\right)^{\nu}$$

$$\sim (\delta q)^{\nu-2}$$

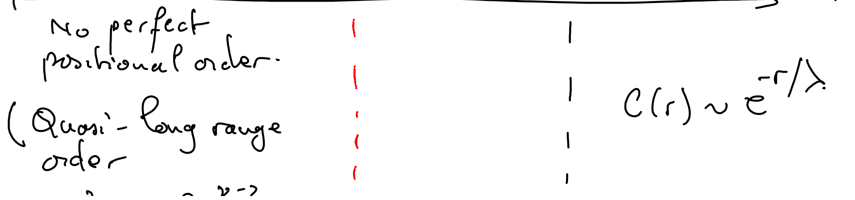
! $\underline{d=2}$ Still $\underline{\text{divergent}}$ Bragg peaks.

Not $\delta(\delta q)$ but power-law divergence $(\nu < 2)$

Quasi-long range order
(power law decay of the
Correlation functions).



$d=3$



$d=2$

T_{melting}.

$d=3$

perfect positional order $\delta(\delta q)$. | Liquid. no positional order. $C(r) \sim e^{-r/\lambda}$ → T

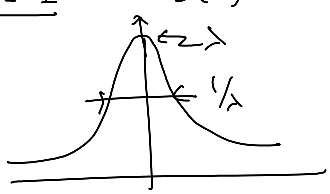
$d=2$

No perfect positional order. (Quasi-long range order $\langle |\,|^2\rangle \sim \delta q^{\,\nu-2}$ | $C(r) \sim e^{-r/\lambda}$ → T

appearance of topological defects (dislocations disclinations etc)

Beresinskii - Kosterlitz - Thouless phase transitions.
(transition <u>without</u> an order parameter)

<u>$d=1$</u> $\quad B(r) \sim \dfrac{T r}{c}$. $\qquad C(r) \sim e^{-\frac{K_0^2 T}{2c} r}$



Lorentzian peaks. $\Rightarrow$ exponential loss of positional order

$$\lambda = \frac{2c}{K_0^2 T}$$

$$e^{-\frac{1}{2}K_0^2 B(r)} \Leftarrow \Theta(1) \qquad \frac{1}{2}K_0^2 B(r) \simeq 1$$

$$B(r) \simeq \frac{2}{K_0^2} \qquad K_0 = \frac{2\pi}{a}. \qquad \Rightarrow B(r) \simeq \frac{2}{(2\pi)^2} a^2 .$$