jeudi 29 octobre 2009
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Quantum Problems

I.] Single particle:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H(t)\,|\psi(t)\rangle$$

$|\psi(t=0)\rangle \qquad |\psi(t)\rangle = U(t,0)\,|\psi(t=0)\rangle$

$$|\psi(t_2)\rangle = U(t_2,t_1)\,|\psi(t_1)\rangle$$

$$i\hbar \frac{\partial}{\partial t}\left[U(t,0)\,|\psi(t=0)\rangle\right] = H(t)\,U(t,0)\,|\psi(t=0)\rangle$$

$$\left[i\hbar \frac{\partial}{\partial t} U(t,0)\right]|\psi(t=0)\rangle = H(t)\,U(t,0)\,|\psi(t=0)\rangle$$

$$\boxed{i\hbar \frac{\partial}{\partial t} U(t,0) = H(t)\,U(t,0)}$$

$H(t) \to H \qquad U(t,0) = e^{-\frac{i}{\hbar}Ht}$

$U(t) \overset{?}{=} e^{-\frac{i}{\hbar}\int_0^t dt' H(t')}$ only true if $H(t_1)$ commute with $H(t_2)$ $\forall\, t_1, t_2$

$$U(0,0) = 1$$
$$U(t_2,t_1) = U(t_2,t_3)\,U(t_3,t_1)$$

$$U(t,0) = 1 + \underset{\lambda}{U_1} + \underset{\lambda^2}{U_2} + \underset{\lambda^3}{U_3} + \underset{\lambda^4}{U_4} + \cdots$$

formally $\quad H(t) = \lambda \tilde{H}(t)$

$$i\hbar \frac{\partial}{\partial t} \underset{\lambda^{n+1}}{U} \leftarrow = \underbrace{H}_{\lambda}\underbrace{U}_{\lambda^n}$$

$$U(t,0) = 1 + U_1 + \cdots$$

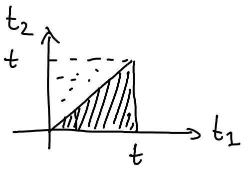
$i\hbar \frac{\partial}{\partial t} U_1(t,0) = H(t)\cdot 1 \qquad U_1(t,0) = \frac{1}{i\hbar}\int_0^t dt'\, H(t')$

$$i\hbar \frac{\partial}{\partial t} U_2(t,0) = H(t)\,\frac{1}{(i\hbar)}\int_0^t dt'\, H(t')$$

$$U_2(t,0) = \frac{1}{(i\hbar)^2}\int_0^t dt_1\, H(t_1)\int_0^{t_1} dt_2\, H(t_2)$$

$$U_3(t,0) = \frac{1}{(i\hbar)^3} \int_0^t dt_1\, H(t_1) \int_0^{t_1} dt_2\, H(t_2) \int_0^{t_2} dt_3\, H(t_3)$$

$$U(t,0) = 1 + \frac{1}{(i\hbar)} \int_0^t dt_1\, H(t_1) + \frac{1}{(i\hbar)^2} \int_0^t dt_1\, H(t_1) \int_0^{t_1} dt_2\, H(t_2) + \dots$$



$$\blacktriangle \quad \int_0^t dt_1\, H(t_1) \int_0^{t_1} dt_2\, H(t_2)$$

$$\overset{?}{=} \frac{1}{2} \int_0^t dt_1\, H(t_1) \int_0^t dt_2\, H(t_2)$$

$$\triangledown \quad \int_0^t dt_1\, H(t_1) \int_{t_1}^t dt_2\, H(t_2)$$

time ordering operator T

$$T\left[O_1(t_1)\, O_2(t_2) \dots O_n(t_n) \right] = O_\alpha(t_\alpha)\, O_\beta(t_\beta) \dots O_\gamma(t_\gamma)$$

$t_1 > t_2$ $\qquad\qquad$ $t_\alpha > t_\beta > t \dots > t_\gamma$

$$T\left[O(t_1)\, O(t_2) \right] = O(t_1)\, O(t_2)$$

$$T\left[O(t_2)\, O(t_1) \right] = O(t_1)\, O(t_2)$$

$$\blacktriangle \quad \int_0^t dt_1\, H(t_1) \int_0^{t_1} dt_2\, H(t_2)$$

$$= \frac{1}{2} T\left[\int_0^t dt_1\, H(t_1) \int_0^t dt_2\, H(t_2) \right]$$

$$U = 1 + \frac{1}{(i\hbar)} T\left[\int_0^t dt_1\, H(t_1) \right] + \frac{1}{(i\hbar)^2} \frac{1}{2!} T\left[\int_0^t dt_1\, H(t_1) \int_0^t dt_2\, H(t_2) \right]$$

$$+ \frac{1}{(i\hbar)^3} \frac{1}{3!} T\left[\int_0^t dt_1\, H(t_1) \int_0^t dt_2\, H(t_2) \int_0^t dt_3\, H(t_3) \right]$$

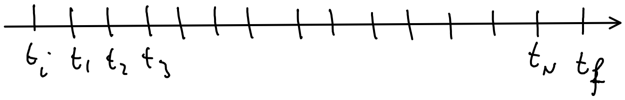
$$+ \dots$$

$$\boxed{U(t,0) = T\left[e^{-\frac{i}{\hbar} \int_0^t dt'\, H(t')} \right]}$$

Compute $U(t,0)$

$$H = H_0[P] + V[R]$$

$$U(t_f, t_i) = U(t_f, t_N)\, U(t_N, t_{N-1})\, U(t_{N-1}, t_{N-2}) \dots U(t_1, t_i)$$



Complete basis. $|x\rangle$ of the 1 particle state

$$\langle x_f | U(t_f, t_i) | x_i \rangle \qquad \forall\, x_f,\, x_i$$

$$= \langle x_f | U(t_f, t_N)\, U(t_N, t_{N-1}) \cdots U(t_1, t_i) | x_i \rangle$$

$$\mathbb{1} = \int dx_N\, |x_N\rangle \langle x_N|$$

$$= \int dx_N\, dx_{N-1}\, dx_{N-2} \cdots dx_1 \quad \langle x_f | U(t_f, t_N) | x_N \rangle \langle x_N | U(t_N, t_{N-1}) | x_{N-1} \rangle \cdots \langle x_1 | U(t_1, t_i) | x_i \rangle$$

$$\langle x_{t_{a+1}} | U(t_{a+1}, t_a) | x_{t_a} \rangle \qquad t_a \approx t_{a+1} \qquad t_{a+1} - t_a \approx \varepsilon$$

$$t_{a+1} \approx t_a \qquad U(t_{a+1}, t_a) \simeq T\left[e^{-\frac{i}{\hbar} H(t_a)(t_{a+1} - t_a)} \right] = e^{-\frac{i}{\hbar} H(t_a)\, \varepsilon}$$

$$e^{-\frac{i}{\hbar}\left[H_0[P] + V[R] \right] \varepsilon} = e^{-\frac{i}{\hbar} H_0[P]\varepsilon}\, e^{-\frac{i}{\hbar} V[R]\varepsilon}\, e^{\varepsilon^2 [H_0(P), V(R)]}$$

in the limit $\varepsilon \to 0$

$$e^{-\frac{i}{\hbar} H(t)\, \varepsilon} = e^{-\frac{i}{\hbar} H_0[P]\varepsilon}\, e^{-\frac{i}{\hbar} V(R)\varepsilon}$$

$|x\rangle$ basis of positions $|r\rangle$

$$R|x\rangle = x|x\rangle$$

$$\langle x_{t_{a+1}} | e^{-\frac{i}{\hbar} H_0(P)\varepsilon}\, e^{-\frac{i}{\hbar} V(R)\varepsilon} | x_{t_a} \rangle$$

$$= e^{-\frac{i}{\hbar} V(x_{t_a})\varepsilon} \langle x_{t_{a+1}} | e^{-\frac{i}{\hbar} H_0[P]\varepsilon} | x_{t_a} \rangle$$

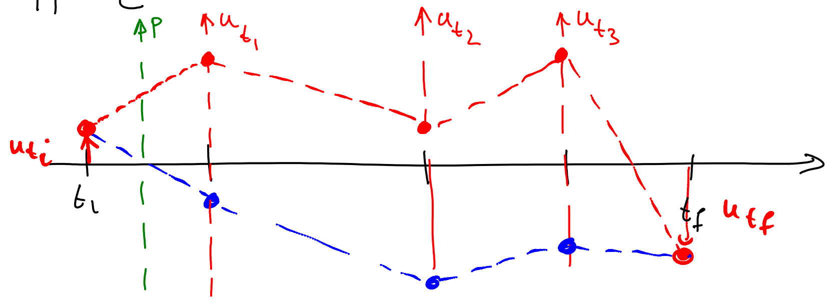
$$= e^{-\frac{i}{\hbar} V(x_{t_a})\varepsilon} \int dp\, \langle x_{t_{a+1}} | e^{-\frac{i}{\hbar} H_0[P]\varepsilon} | p \rangle \langle p | x_{t_a} \rangle$$

$$= e^{-\frac{i}{\hbar} V(x_{t_a})\varepsilon} \int dp\, \underbrace{\langle x_{t_{a+1}} | p \rangle}_{\frac{1}{(2\pi)^{d/2}} e^{\frac{ip}{\hbar} x_{t_{a+1}}}} \underbrace{\langle p | x_{t_a} \rangle}_{\frac{1}{(2\pi)^{d/2}} e^{-\frac{ip}{\hbar} x_{t_a}}} e^{-\frac{i}{\hbar} H_0[p]\varepsilon}$$

$$= \int \frac{dp}{(2\pi)^d} \; e^{-\frac{i\varepsilon}{\hbar}\left[H_0[p] + V(u_{t_a})\right]} \qquad e^{\frac{ip}{\hbar}\left[u_{t_{a+1}} - u_{t_a}\right]}$$

$$\langle u_f | U(t_f, t_i) | u_i \rangle = \int du_N \, du_{N-1} \, du_{N-2} \cdots du_1 \int \frac{dp_N}{(2\pi)^d} \int \frac{dp_{N-1}}{(2\pi)^d} \cdots\cdots$$

$$\prod e^{\frac{ip}{\hbar}\left[u_{t_f} - u_{t_N}\right] - \frac{i\varepsilon}{\hbar}\left[H_0[p_N] + V(u_{t_N})\right]} \quad \cdots$$



$$u_{t_{a+1}} - u_{t_a} \approx (t_{a+1} - t_a) \left.\frac{\partial u(t)}{\partial t}\right|_{t_a} = \varepsilon \left.\frac{\partial u(t)}{\partial t}\right|_{t_a}$$

$$e^{\frac{i}{\hbar}\left[\varepsilon \left.\frac{\partial u}{\partial t}\right|_{t_N}\right] - \frac{i}{\hbar}\varepsilon\left[H_0[p_N] + V(u_{t_N})\right]}$$

$$\times \; e^{\frac{i}{\hbar}p_N\left[\varepsilon \left.\frac{\partial u}{\partial t}\right|_{t_{N-1}}\right] - \frac{i}{\hbar}\varepsilon\left[H_0[p_{N-1}] + V(u_{t_{N-1}})\right]}$$

$$= \; e^{\int_{t_i}^{t_f} dt \cdot \left[\frac{i}{\hbar} p(t) \left(\frac{\partial u}{\partial t}\right) - \frac{i}{\hbar}\left[H_0[p(t)] + V[u(t)]\right]\right]}$$

$$\langle u_{t_f} | U(t_f, t_i) | u_{t_i} \rangle = \int_{u(t_i) = u_{t_i}}^{u[t_f] = u_{t_f}} \mathcal{D}u[t] \int \mathcal{D}p[t] \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \left[p(t)\frac{\partial u}{\partial t} - \left[H_0[p(t)] + V(u(t))\right]\right]}$$

$$= \int_{u(t_i) = u_{t_i}}^{u[t_f] = u_{t_f}} \mathcal{D}u[t] \int \mathcal{D}p[t] \qquad e^{\frac{i}{\hbar} \underbrace{\int_{t_i}^{t_f} dt \; \mathcal{L}[t]}_{S(t_f,\, t_i)}}$$

$$\mathcal{L}(t) = p(t)\frac{\partial u}{\partial t} - H[p(t), u(t)]$$

$\hbar \to 0$ $\left\{ \begin{array}{l} \text{Saddle point} \\ 1 \text{ trajectory.} \end{array} \right.$ $\Rightarrow$ Classical physics.

$$H_0[p] = \frac{p^2}{2m}.$$

$$\int \mathcal{D}p[t] \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \left[\underbrace{p(t)\frac{\partial u}{\partial t} - \frac{p^2}{2m}} - V(u(t))\right]}$$

$$\int \mathcal{D}p[t] \; e^{\frac{i}{\hbar}\int_{t_i}}$$

$$-\frac{1}{2m}\left[p - \left(\frac{\partial u}{\partial t}\right)m\right]^2 + \left(\frac{\partial u}{\partial t}\right)^2 \frac{m}{2}.$$

$$\int \mathcal{D}p[t] \; e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \; \frac{1}{2m}\Big[\underbrace{p(t) - m\frac{\partial u}{\partial t}}_{\tilde{p}(t)}\Big]^2} \qquad e^{\frac{i}{\hbar}\int_{t_i}^{t_f}\left[\frac{m}{2}\left(\frac{\partial u}{\partial t}\right)^2 - V(u(t))\right]}$$

$$= \text{Cste} \quad e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \left[\frac{m}{2}\left(\frac{\partial u}{\partial t}\right)^2 - V(u(t))\right]}$$

$$\langle u_{t_f} | U(t_f, t_i) | u_{t_i}\rangle = \int_{u[t_i]=u_{t_i}}^{u[t_f]=u_{t_f}} \mathcal{D}u[t] \quad e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\left[\frac{m}{2}\left(\frac{\partial u}{\partial t}\right)^2 - V(u(t))\right]}$$

Finite temperature:

density matrix $\qquad \rho = \frac{1}{Z} e^{-\beta H} \qquad =$

- H independent of time $\qquad H|\psi_n\rangle = E_n|\psi_n\rangle$

$$\rho = \sum_{n=0}^{+\infty} \frac{1}{Z} e^{-\beta E_n} |\psi_n\rangle\langle\psi_n| \qquad\qquad \sum_{n=0}^{+\infty} e^{-\beta E_n} = Z$$

$$\langle O\rangle = \mathrm{Tr}[\rho\, O]$$

$$Z = \mathrm{Tr}[\rho] = \mathrm{Tr}\left[e^{-\beta H}\right] = \int du \; \langle u | e^{-\beta H} | u\rangle$$

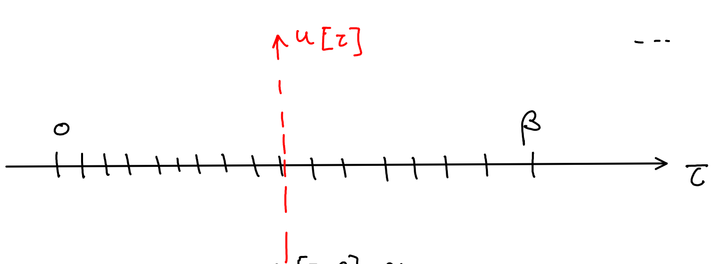
time evolution. $\quad U(t_1, t_2) = e^{-\frac{i}{\hbar}H(t_2 - t_1)}$

finite temperature $\qquad e^{-\beta H}$

"imaginary time"

$$Z = \int du_0 \int du_N\, du_{N-1}\, du_{N-2} \cdots du_1 \quad \langle u_0| e^{-\frac{\beta}{N}H} |u_N\rangle \langle u_N| e^{-\frac{\beta}{N}H}|u_{N-1}\rangle$$

$$\cdots \quad \langle u_1 | e^{-\frac{\beta}{N}\cancel{H}} | u_0\rangle$$



$$\tau \in [0, \beta]$$

$$Z = \int du_0 \int_{u[\tau=0]=u_0}^{u[\tau=\beta]=u_0} \mathcal{D}u[\tau] \int \mathcal{D}p[\tau] \quad e^{\int_0^\beta d\tau \left[\frac{i}{\hbar}p\left(\frac{\partial u}{\partial \tau}\right) - H[p,u]\right]}$$

$$\int_0^\beta d\tau \left[\frac{i}{\hbar}\, p(\tau)\frac{\partial u}{\partial \tau} - H_0[p(\tau)] - V[u(\tau)]\right]$$

$it \to \tau$

$$\int_{t_1}^{t_2} dt \left[\frac{i}{\hbar}\, p(t) \frac{\partial u}{\partial t} - \frac{i}{\hbar}\Big(\underbrace{H_0[p(t)]}_{\frac{p^2}{2m}} + V[u(t)] \Big) \right]$$

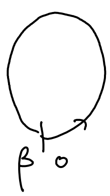
$$\int_0^\beta d\tau \left[-\frac{p^2}{2m} + i p \dot{u} \right]$$

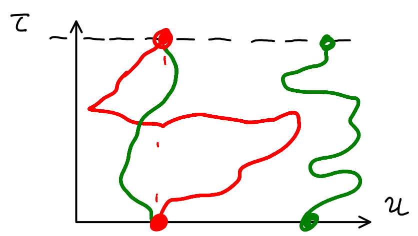
$$= \int_0^\beta d\tau \left[-\frac{(p - i\dot{u} m)^2}{2m} - \frac{m}{2} (\dot{u})^2 \right]$$

$$\text{cste} \quad e^{-\int_0^\beta d\tau \, \frac{m}{2} \left(\frac{\partial u}{\partial \tau}\right)^2}$$

$$Z = \int du_0 \int_{u[\tau=0]=u_0}^{u[\tau=\beta]=u_0} \mathcal{D}u[\tau] \; e^{-\int_0^\beta d\tau \left[\frac{m}{2} \left(\frac{\partial u}{\partial \tau}\right)^2 + V(u(\tau)) \right]}$$

$$= \int_{\text{periodic}} \mathcal{D}u[\tau] \; e^{-\frac{1}{\hbar}\int_0^\beta d\tau \left[\frac{m}{2} \left(\frac{\partial u}{\partial \tau}\right)^2 + V(u(\tau)) \right]}$$





$$Z_{\text{classical problem}} = \sum_{\text{Conf.}} e^{-\frac{1}{T_{cl.}} H[C]_{cl..}}$$

Quantum problem $\longleftrightarrow$ Classical problem.

$u \to d$ dimensions $\qquad d + \text{"}1\text{"}$

$u_{\tau=0} \in [-\infty, +\infty]$
$u_{\tau=1}$ $\longleftrightarrow$
$u_{\tau=2}$

$\hbar \longleftrightarrow T_{cl.}$

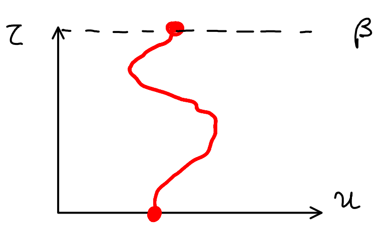
$$\int_0^\beta \frac{m}{2}\left(\frac{\partial u}{\partial \tau}\right)^2 + V[u(\tau)] \, d\tau \longleftrightarrow H_{cl.}$$

<u>Example</u> $\qquad ^2$

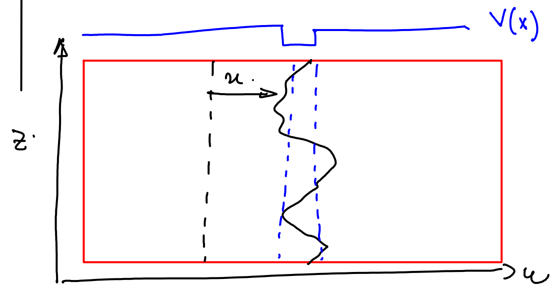
$$\Big| \quad Z = \int \mathcal{D}u[z] \; e^{-\frac{c}{2T_{cl}} \int dz \, (\nabla_z u)^2}$$

$$H = \frac{p^2}{2m}$$

$$\int_{\mathcal{B}} \mathcal{D}u[\tau]\, e^{-\frac{1}{\hbar}\int_0^\beta d\tau \frac{m}{2}\left(\frac{\partial u}{\partial \tau}\right)^2}$$



$$H_{el} = \frac{c}{2}\int dz \left(\nabla_z u\right)^2$$



$u(z)$

Free quantum particle (1 dimension)	$\longleftrightarrow$	Line in a $d=2$ space
$\frac{m}{2\hbar}$	$\longleftrightarrow$	$\frac{c}{T_{clas}}$.
β	$\longleftrightarrow$	finite size L_z

Classical problem : phase transition. ‖ Quantum phase transitions

T_c. → "$\hbar$" → one of the parameters in the system.

$L_z \to \infty$ → $\beta = \infty$ $\quad T_Q = 0$

2.) Correlation functions

$$Z = \int \mathcal{D}u[\tau]\, e^{-S} \qquad S = \int_0^\beta d\tau \left[\, ----- \,\right]$$

$$\langle O(\tau_1)\, O(\tau_2) \ldots O(\tau_p) \rangle = \frac{1}{Z}\int \mathcal{D}u[\tau]\; O(u(\tau_1))\, O(u(\tau_2)) \ldots e^{-S}.$$

$$B(r) = \langle [u(r) - u(0)]^2 \rangle \qquad \langle (u(\tau) - u(0))^2 \rangle$$

Evolution in time ?.

$$O_1 \,|\psi(t_1)\rangle = O_1\, U(t_1, 0)\, |\psi(t=0)\rangle$$

$$U(t_2, t_1)\, O_1\, |\psi(t_1)\rangle$$

$$U(t_2, t_1)\, O_1\, |\psi(t_1)\rangle$$

$$O_2\, U(t_2, t_1)\, O_1\, |\psi(t_1)\rangle = |\psi_{①}(t_2)\rangle$$

$$U(t_2, 0)\, |\psi(t=0)\rangle = |\psi_{②}(t_2)\rangle$$

$$\langle \psi_{②}(t_2) | \psi_{①}(t_2)\rangle = \langle \psi(t=0)| \underbrace{U^{+}(t_2, 0)}_{U(0,t_2)}\, O_2\, U(t_2, t_1)\, O_1\, U(t_1, 0)|\psi(t=0)\rangle$$

H independent of time.

$$U(t_2, t_1) = e^{-iH(t_2 - t_1)}$$

$$\langle \psi(t=0)|\, e^{iHt_2}\, O_2\, e^{-iH(t_2-t_1)}\, O_1\, e^{-iHt_1}\, |\psi(t=0)\rangle$$

$$\hat{O}(t) = e^{iHt}\, O\, e^{-iHt}$$ Heisenberg representation

$$\langle \psi(t=0)|\, \hat{O}_2(t_2)\, \hat{O}_1(t_1)\, |\psi(t=0)\rangle$$

$$\downarrow$$

$$\mathrm{Tr}\left[\rho\, \hat{O}_2(t_2)\, \hat{O}_1(t_1)\right] = \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}\, \hat{O}_2(t_2)\, \hat{O}_1(t_1)\right]$$

physical object that we need to compute

$$\mathrm{Tr}\left[e^{-\beta H}\, e^{iHt_2}\, O_2\, e^{-iHt_2} \cdots\right.$$

New Non physical object.

$$\hat{O}(\tau) = e^{H\tau}\, O\, e^{-H\tau}. \qquad \hat{O}(\tau)^{+} \neq \hat{O}^{+}(\tau)$$

$$\langle \hat{O}_2(\tau_2)\, \hat{O}_1(\tau_1)\rangle = \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}\, \hat{O}_2(\tau_2)\, \hat{O}_1(\tau_1)\right]$$

$$= \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}\, e^{H\tau_2}\, O_2\, e^{-H(\tau_2-\tau_1)}\, O_1\, e^{-H\tau_1}\right]$$

$$\langle \hat{O}_2(\tau_2-\tau_1)\, \hat{O}_1(0)\rangle = \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}\, e^{H(\tau_2-\tau_1)}\, O_2\, e^{-H(\tau_2-\tau_1)}\, O_1\right]$$

$$\langle \hat{O}_2(\tau)\, \hat{O}_1(0)\rangle = \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}\, e^{H\tau}\, O_2\, e^{-H\tau}\, O_1\right]$$

$$\langle \hat{O}_2(\tau+\beta)\, \hat{O}_1(0)\rangle = \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}\, e^{H(\tau+\beta)}\, O_2\, e^{-H(\tau+\beta)}\, O_1\right]$$

$$= \frac{1}{Z} \mathrm{Tr}\left[e^{H\tau} O_2 e^{-H(\tau+\beta)} O_1 \right]$$

$$= \frac{1}{Z} \mathrm{Tr}\left[e^{-H(\tau+\beta)} O_1 e^{H\tau} O_2 \right]$$

$$= \frac{1}{Z} \mathrm{Tr}\left[e^{-\beta H} O_1 e^{H\tau} O_2 e^{-H\tau} \right]$$

$$= \langle \hat{O}_1(0) \hat{O}_2(\tau) \rangle$$

$$\langle T_\tau \hat{O}_2(\tau_2) \hat{O}_1(\tau_1) \dots \rangle$$

T_τ : "imaginary time" ordering operator

$$\tau_1 > \tau_2 \qquad T[O(\tau_1) O(\tau_2)] = O(\tau_1) O(\tau_2)$$

$$\tau_1 < \tau_2 \qquad T[\qquad] = O(\tau_2) O(\tau_1).$$

$$\langle T_\tau \hat{O}_2(\tau+\beta) \hat{O}_1(0) \rangle = \langle T_\tau \hat{O}_2(\tau) \hat{O}_1(\beta) \rangle$$

periodic in τ of period β.

"Time" ordered correlation functions

↳ this object will have a good path integral representation.

$$\langle T_\tau \hat{O}_2(\tau_2) \hat{O}_1(\tau_1) \rangle = \frac{1}{Z} \mathrm{Tr}\left[e^{-\beta H} T_\tau (\qquad) \right]$$

$$\tau_2 > \tau_1 \qquad \longrightarrow \frac{1}{Z} \mathrm{Tr}\left[e^{-\beta H} \hat{O}_2(\tau_2) \hat{O}_1(\tau_1) \right]$$

$$e^{-\beta H} \equiv e^{-\int_0^\beta d\tau \hat{H}(\tau)}$$

$$\hat{H}(\tau) = e^{H\tau} H e^{-H\tau}$$

$$\langle T_\tau \hat{O}_2(\tau_2) \hat{O}_1(\tau_1) \rangle = \mathrm{Tr}\left[e^{-\int_{\tau_2}^{\beta} d\tau' \hat{H}(\tau')} \hat{O}(\tau_2) e^{-\int_{\tau_1}^{\tau_2} d\tau' \hat{H}(\tau')} \hat{O}(\tau_1) e^{-\int_0^{\tau_1} d\tau' \hat{H}(\tau')} \right]$$

$$\rightarrow \int du_0 \dots du_{\tau_1} du'_{\tau_1} du_{\tau_2} du'_{\tau_2} \quad \langle u_0 | e^{-(\beta-\tau_2)H} | u_{\tau_2} \rangle \langle u_{\tau_2} | O(\tau_2) | u'(\tau_2) \rangle$$

$$\langle u'(\tau_2) | e^{-H(\tau_2-\tau_1)} | u(\tau_1) \rangle \langle u(\tau_1) | \hat{O}(\tau_1) | u'(\tau_1) \rangle$$

$$\langle u'(\tau_1) | e^{-\tau_1 H} | u_0 \rangle$$

operators that are diagonal in the basis u

$$\langle x_{\tau_2} | O(\tau_2) | x'_{\tau_2} \rangle = \mathcal{O}[x(\tau_2)] \, \delta_{x,x'}$$

$$\to \int dx_0 \, dx_1 \, dx_2 \quad \langle x_0 | e^{-(\beta-\tau_2)H} | x_2 \rangle \, \mathcal{O}_2(x_2)$$
$$\langle x_2 | e^{-(\tau_2-\tau_1)H} | x_1 \rangle \, \mathcal{O}_1(x_1)$$
$$\langle x_1 | e^{-\tau_1 H} | x_0 \rangle$$

$$\frac{1}{Z} \int_{\text{periodic}} \mathcal{D}x[\tau] \; e^{-S} \; \mathcal{O}_2(x(\tau_2)) \, \mathcal{O}_1(x(\tau_1))$$

$$= \frac{1}{Z} \mathrm{Tr}\left[e^{-\beta H} \, T_\tau \; \hat{O}_2(\tau_2) \, \hat{O}_1(\tau_1) \right]$$

$$\hat{O}(\tau) = e^{H\tau} \, O \, e^{-H\tau}.$$

Properties of time ordered correlations.

$$G(\tau) = \langle T_\tau \; \hat{O}_2(\tau) \, \hat{O}_1(0) \rangle$$

$$G(\tau+\beta) = G(\tau) \Rightarrow$$

only contains $\omega_n = \frac{2\pi}{\beta} n$.

$$\begin{cases} G(\tau) = \frac{1}{\beta} \sum_n e^{-i\omega_n \tau} \, G(i\omega_n) \\ G(i\omega_n) = \int_0^\beta d\tau \; e^{i\omega_n \tau} \, G(\tau) \end{cases} \qquad e^{-i\omega_n \beta} = 1$$

Matsubara frequencies.

Special case of $\omega_n = 0$

$$H_0 + \int dx \; h(x) \, \mathcal{O}$$

$$Z = \int \mathcal{D}x[\tau] \; e^{-\int_0^\beta d\tau \left[\left(\frac{m}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 - - - - - \right)_{H_0} + \int dx \, h(x) \, \mathcal{O}[x(\tau)] \right]}$$

$$\int_0^\beta d\tau \int dx \; h(x) \, \mathcal{O}_x[x(\tau)] = \int dx \; h(x) \, \mathcal{O}[x, \omega_n = 0]$$

$$\mathcal{O}(\omega_n) = \int_0^\beta d\tau \; e^{i\omega_n \tau} \, \mathcal{O}[x(\tau)].$$

$$F = -\frac{1}{\beta} \log Z. \qquad \frac{\partial F}{\partial h(x)} = -\frac{1}{\beta} \frac{1}{Z} \frac{\partial Z}{\partial h(x)}$$

$$- \int_0^\beta d\tau \, (\quad) - \int_0^\beta d\tau \int dx \, h(x) \, \mathcal{O}(x, x(\tau))$$

$$F = -\frac{1}{\beta} \log Z . \qquad \frac{\partial F}{\partial h(x)} = -\frac{1}{\beta} \frac{1}{Z} \frac{\partial Z}{\partial h(x)}$$

$$\frac{\partial Z}{\partial h(x)} = \frac{\partial}{\partial h(x)} \int \mathcal{D}u[\tau] \; e^{-\int_0^\beta d\tau \, (\quad)_{H_0} - \int_0^\beta d\tau \int dx \, h(x)\, \mathcal{O}(x, u(\tau))}$$

$$= \int \mathcal{D}u[\tau] \; e^{-\int_0^\beta d\tau \, (\quad)_{H_0}} \left[\int_0^\beta d\tau \, \mathcal{O}(x, u(\tau)) \right] e^{-\int_0^\beta d\tau \cdots}$$

$$\left. \frac{\partial Z}{\partial h(x)} \right|_{h=0} = \int \mathcal{D}u[\tau] \left[\int_0^\beta d\tau \, \mathcal{O}(x, u(\tau)) \right] e^{-S_0}$$

$$\frac{\partial F}{\partial h(x)} = \frac{-1}{\beta} \int \mathcal{D}u \frac{1}{Z_0} \left[\int_0^\beta d\tau \; \mathcal{O}(x, u(\tau)) \right] e^{-S_0} .$$

$$= -\frac{1}{\beta} \left\langle \int_0^\beta d\tau \; \mathcal{O}(x, u(\tau)) \right\rangle_{H_0}$$

$$= -\frac{1}{\beta} \left\langle \mathcal{O}(x, \omega_n = 0) \right\rangle_{H_0}$$

III] <u>Linear response</u>

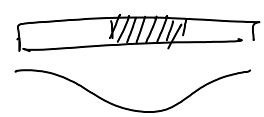
$$H = H_0 + H_{pert}(t)$$

H_0 ↳ thermodynamic equilibrium (time independant)

$$H_{pert}(t) = \int dr \; \lambda(r, t) \; \mathcal{O}(r)$$

<u>examples</u>

	$\mathcal{O}$
$A(x,t)$	J
$h(x,t)$	m .
$\mu(x,t)$	$\rho(x)$



$$\langle A(x_0) \rangle_{t_0} . \qquad \langle \psi(t_0) | A(x_0) | \psi(t_0) \rangle$$

$$\langle A(x_0) \rangle_{t_0} = \left[\langle A(x_0) \rangle_{t_0} \right]_{H_0} + \int dx\,dt \; \underbrace{\chi(x_0, x; t_0, t)}_{\text{can be computed from } H_0 \text{ and } \mathcal{O}, A} \; \lambda(x,t) + \cdots \lambda^2$$

$$\chi(x_0, x; t_0 - t)$$

<u>$T=0$</u> $\quad \langle A(x_0) \rangle_{t_0} \longrightarrow \langle \psi(t_0) | A(x_0) | \psi(t_0) \rangle$

$\underline{T \neq 0}$ $\qquad \langle A(x_0) \rangle_{t_0}$

if H is time independant $\qquad \frac{1}{Z} \mathrm{Tr}\left[e^{-\beta H} A \right] = \mathrm{Tr}\left[\rho A \right]$

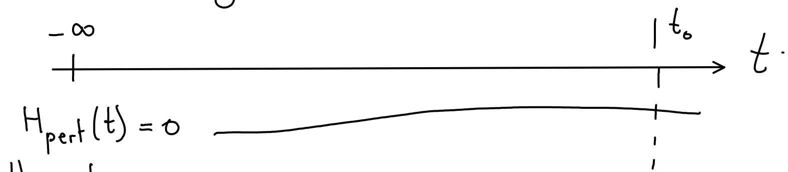
H is time dependant.

$$\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \qquad p_1 \ 1, 0000$$

$$\hookrightarrow \quad |\psi_1\rangle \langle \psi_1| \quad \longrightarrow \quad |\psi(t)\rangle \langle \psi(t)|$$

$$\langle A(x_0) \rangle_{t_0} = \mathrm{Tr}\left[\rho(t_0) A(x_0) \right]$$

what is $\rho(t_0)$ when H which is time dependant.



thermodynamic eq.

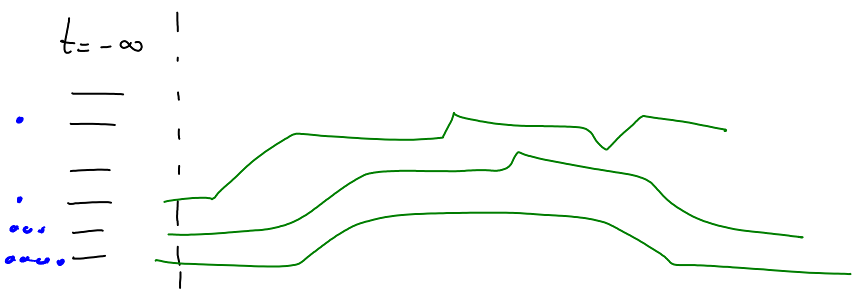
$$\rho = \frac{1}{Z_0} e^{-\beta H_0}$$

$$\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$$

$$\frac{\partial \rho}{\partial t} = \sum_i \frac{\partial p_i}{\partial t} |\psi_i\rangle \langle \psi_i| + \sum_i p_i \left(\frac{\partial}{\partial t} |\psi_i\rangle \right) \langle \psi_i| + \sum_i p_i |\psi_i\rangle \left(\frac{\partial}{\partial t} \langle \psi_i| \right)$$

Assumption: the perturbation is applied adiabatiquely

$\Rightarrow$ p_i are $\underline{\text{not}}$ changed by the perturbation.



$$\frac{\partial \rho}{\partial t} = \sum_i p_i \left(\frac{\partial}{\partial t} |\psi_i\rangle \right) \langle \psi_i| + p_i |\psi_i\rangle \left(\frac{\partial}{\partial t} \langle \psi_i| \right)$$

$$\begin{cases} i \partial_t |\psi_i(t)\rangle = H(t) |\psi_i(t)\rangle \\ -i \partial_t \langle \psi_i(t)| = \langle \psi_i(t)| H(t) \end{cases}$$

$$\frac{\partial \rho}{\quad} = \sum p_i \ \ H(t) \ |\psi_i(t)\rangle \langle \psi_i(t)|$$

$$\frac{\partial \rho}{\partial t} = \sum_i p_i \frac{1}{i} H(t)\,|\psi_i(t)\rangle\langle\psi_i(t)| - \sum_i p_i \frac{1}{i} |\psi_i(t)\rangle\langle\psi_i(t)|\, H(t)$$

$$\boxed{\frac{\partial \rho}{\partial t} = -i\left[H(t), \rho(t)\right]}$$

$$\begin{cases} H = H_0 + H_{pert}(t) \\ \rho = \rho_0 + \delta\rho(t) + \ldots\ldots \end{cases} \qquad \rho_0 = \frac{1}{Z_0} e^{-\beta H_0}$$

$$\frac{\partial \delta\rho(t)}{\partial t} = -i\left[H_0 + H_{pert}(t), \rho_0 + \delta\rho(t)\right]$$

$$= -i[H_0, \rho_0] - i[H_0, \delta\rho(t)] - i[H_{pert}(t), \rho_0] - i\cancel{[H_{pert}(t), \delta\rho(t)]}$$

$$\frac{\partial \delta\rho(t)}{\partial t} = -i[H_0, \delta\rho(t)] - i[H_{pert}(t), \rho_0]$$

$$\frac{\partial}{\partial t}\left(e^{iH_0t}\delta\rho(t)e^{-iH_0t}\right) = e^{iH_0t}\frac{\partial\delta\rho}{\partial t}e^{-iH_0t} + e^{iH_0t}(iH_0)\delta\rho\, e^{-iH_0t} + e^{iH_0t}\delta\rho(-iH_0)e^{-iH_0t}$$

$$e^{-iH_0t}\left[\frac{\partial}{\partial t}\left(e^{iH_0t}\delta\rho(t)e^{-iH_0t}\right)\right]e^{iH_0t} = \frac{\partial\rho}{\partial t} + i[H_0, \delta\rho]$$

$$e^{-iH_0t}\left[\frac{\partial}{\partial t}\left(e^{iH_0t}\delta\rho\, e^{-iH_0t}\right)\right]e^{+iH_0t} = -i[H_{pert}(t), \rho_0]$$

$$\frac{\partial}{\partial t}\left(e^{iH_0t}\delta\rho(t)e^{-iH_0t}\right) = -i\, e^{iH_0t}[H_{pert}(t), \rho_0]\,e^{-iH_0t}$$

$$H_{pert}(t) = \int dr\, \lambda(r,t)\, \mathcal{O}(r)$$

$$e^{iH_0t}\delta\rho(t)e^{-iH_0t} = -i\int_{-\infty}^{t} dt'\, e^{iH_0t'}[H_{pert}(t'), \rho_0]\,e^{-iH_0t'} + \underbrace{C}_{=0}$$

$$e^{iH_0t}\delta\rho(t)e^{-iH_0t} = -i\int_{-\infty}^{t} dt'\, e^{iH_0t'}[H_{pert}(t'), \rho_0]\,e^{-iH_0t'}$$

$$\delta\rho(t) = -i\, e^{-iH_0t}\left[\int^{t} dt'\, e^{iH_0t'}[H_{pert}(t'), \rho_0]\,e^{-iH_0t'}\right]e^{iH_0t}$$

$$\delta\rho(t) = -i\, e^{-iH_0 t}\left[\int_{-\infty}^{t} dt'\, e^{iH_0 t'}\left[H_{pert}(t'),\rho_0\right]e^{-iH_0 t'}\right]e^{iH_0 t}$$

$$H_{pert}(t') = \int dr\, \lambda(r,t')\, O$$

$$e^{iH_0t'}\left[H_{pert}(t'),\rho_0\right]e^{-iH_0t'} = \left[e^{iH_0t'}H_{pert}(t')\,e^{-iH_0t'},\rho_0\right]$$

$$= \int dr\,\lambda(r,t')\,\left[\hat{O}(t'),\rho_0\right]$$

$$\delta\rho(t) = -i\, e^{-iH_0t}\left[\int_{-\infty}^{t} dt' \int dr\, \lambda(r,t')\left[\hat{O}(r,t'),\rho_0\right]\right]e^{+iH_0t}$$

$$\langle A(x_0)\rangle_{t_0} = \mathrm{Tr}\left[\rho(t_0)\, A(x_0)\right]$$

$$= \mathrm{Tr}\left[\rho_0\, A(x_0)\right] + \mathrm{Tr}\left[\delta\rho(t_0)\, A(x_0)\right]$$

$$\mathrm{Tr}\left[\delta\rho(t_0)A(x_0)\right] = -i\,\mathrm{Tr}\left[e^{-iH_0t_0}\left(\qquad\right)e^{+iH_0t_0}\, A(x_0)\right]$$

$$= -i\,\mathrm{Tr}\left[\left(\qquad\right)e^{iH_0t_0}A(x_0)\,e^{-iH_0t_0}\right] = -i\,\mathrm{Tr}\left[\left(\quad\right)\hat{A}(x_0,t_0)\right]$$

$$= -i\,\mathrm{Tr}\left[\int_{-\infty}^{t_0} dt \int dr\, \lambda(r,t)\left[\hat{O}(r,t),\rho_0\right]\hat{A}(x_0,t_0)\right]$$

$$= -i\int_{-\infty}^{t_0} dt \int dr\, \lambda(r,t)\quad \mathrm{Tr}\left[\left[\hat{O}(r,t),\rho_0\right]\hat{A}(x_0,t_0)\right]$$

$$\mathrm{Tr}\left[[A,B]C\right] = \mathrm{Tr}\left[ABC - BAC\right] = \mathrm{Tr}\left[BCA - BAC\right] = \mathrm{Tr}\left[B[C,A]\right]$$

$$\langle A(x_0)\rangle_{t_0} = \langle A(x_0)\rangle_{H_0} - i\int_{-\infty}^{t_0} dt\int dr\,\lambda(r,t)\ \mathrm{Tr}\left[\rho_0\left[\hat{A}(x_0,t_0),\hat{O}(r,t)\right]\right]$$

$$= \langle A(x_0)\rangle_{H_0} - i\int_{-\infty}^{t_0} dt\int dr\,\lambda(r,t)\ \left\langle\left[\hat{A}(x_0,t_0),\hat{O}(r,t)\right]\right\rangle_{H_0}$$

$$\boxed{\chi(r_0,r,t_0-t) = -i\,\theta(t_0-t)\left\langle\left[\hat{A}(x_0,t_0),\hat{O}(x,t)\right]\right\rangle_{H_0}}$$
(retarded correlation function)

$$\langle A(r_0,t_0)\rangle = \int dr\,dt\ \chi(r_0,r,t_0-t)\,\lambda(r,t)$$

translational invariance in H_0 $\quad \chi(r_0-r,\ t_0-t)$

$$A(k,\omega) = \int dr\,dt\ e^{-i(kr-\omega t)}\langle A(r,t)\rangle$$

$$A(k,\omega) = \chi(k,\omega)\,\lambda(k,\omega)$$

$$\chi(k,\omega) = \int dr\,dt\; e^{-i(kr-\omega t)}\; \chi(r,t).$$

$$\chi(\omega) = -i\int dt\; e^{i\omega t}\; \theta(t)\; \langle [\hat{A}(t), \hat{O}(0)] \rangle$$

$$= -i\int_0^{+\infty} dt\; e^{i\omega t}\; \langle [\hat{A}(t), \hat{O}(0)] \rangle$$

Regularize the integral

$$\chi(\omega) = -i\int_0^{+\infty} dt\; e^{i\omega t}\; e^{-\delta t}\; \langle \qquad \rangle \qquad \delta > 0 \to 0$$

ensures that the integral converges at $t \to +\infty$

$$\int_0^{+\infty} dt\; e^{i(\omega+i\delta)t} \; --- . \qquad \omega \to \omega + i\delta$$

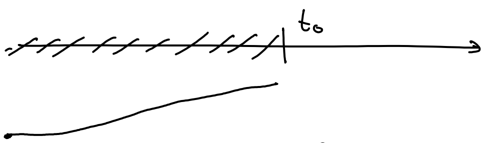
$$A(t_0) = \int dt\; \chi(t_0 - t)\; \lambda(t).$$

$$\lambda(t) = e^{i\omega t}.$$

$$A(t_0) = \int dt\; \chi(t_0 - t)\; e^{i\omega t}\; e^{i(\omega_0 t_0 - \omega_0 t_0)}$$

$$= e^{i\omega t_0} \int dt\; \chi(t_0 - t)\; e^{-i\omega(t_0 - t)}$$

One needs $\lambda(t \to -\infty) \to 0$ $\qquad \lambda(t) \equiv e^{i\omega t}\; e^{\delta t}$



$\delta \to 0$

$$A(t_0) = \int dt\; \chi(t_0 - t)\; e^{i\omega t}\; e^{\delta t}.$$

$$= e^{i\omega t_0}\; e^{\delta t_0}. \underbrace{\int dt\; \chi(t_0 - t)\; e^{i\omega(t - t_0)}\; e^{\delta(t - t_0)}}$$

$$\int dt\; \chi(t_0 - t)\; e^{(i\omega + \delta)(t - t_0)}$$

$$\chi(t_0 - t) = -i\,\theta(t_0 - t)\; \langle [\hat{A}(t_0), \hat{O}(t)] \rangle$$

$$\chi_{ret}(\omega) = -i\int_0^{+\infty} dt\; e^{i(\omega + i\delta)t}\; \langle [\hat{A}(t), \hat{O}(0)] \rangle$$

Lehmann representation.

$$H\,|n\rangle = E_n\,|n\rangle$$

$$\langle [\hat{A}(t), O(0)] \rangle = \frac{1}{Z}\sum_n \langle n | e^{-\beta H} \; e^{iHt} A e^{-iHt} \underset{\sum_m |m\rangle\langle m|}{\uparrow} O - \underset{\uparrow}{O} (e^{iHt} A e^{-iHt}) | n \rangle$$

$$= \frac{1}{Z}\sum_{nm} \Big[e^{-\beta E_n} \; e^{iE_n t} \; e^{-iE_m t} \; \langle n|A|m\rangle \langle m|O|n\rangle$$

$$- \; e^{-\beta E_n} \; e^{iE_m t} \; e^{-iE_n t} \langle n|O|m\rangle \langle m|A|n\rangle \Big]$$

$$= \frac{1}{Z}\sum_{n,m} e^{-\beta E_n} \; e^{i(E_n - E_m)t} \quad \langle n|A|m\rangle \langle m|O|n\rangle$$

$$e^{-\beta E_m} \; e^{i(E_n - E_m)t} \quad \langle n|A|m\rangle \langle m|O|n\rangle$$

$$= \frac{1}{Z}\sum_{n,m} \langle n|A|m\rangle\langle m|O|n\rangle \; e^{i(E_n - E_m)t} \quad \left(e^{-\beta E_n} - e^{-\beta E_m} \right)$$

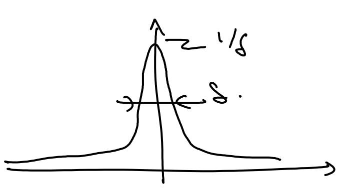
$$-i \int_0^{+\infty} dt \; e^{i(\omega + i\delta)t} \; e^{i(E_n - E_m)t} = -i \; \frac{0 \quad - 1}{i\,[\omega + i\delta + E_n - E_m]}$$

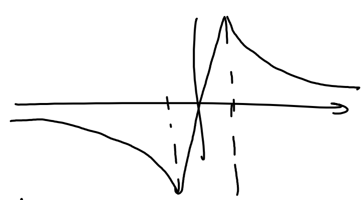
$$= \frac{1}{\omega + E_n - E_m + i\delta}$$

$$\chi_{ref}(\omega) = \frac{1}{Z}\sum_{n,m} \langle n|A|m\rangle\langle m|O|n\rangle \; \frac{e^{-\beta E_n} - e^{-\beta E_m}}{\omega + E_n - E_m + i\delta}$$

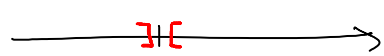
$$\lim_{\delta \to 0} \frac{1}{x + i\delta} = P\left(\frac{1}{x}\right) - i\pi\,\delta(x)$$

$$\frac{1}{x + i\delta} = \frac{x - i\delta}{x^2 + \delta^2} \qquad \begin{cases} \dfrac{x}{x^2+\delta^2} \\[2ex] \dfrac{\delta}{x^2+\delta^2} \end{cases}$$





$$\int_a^b dx \; P\left(\frac{1}{x}\right) \rightarrow \int_a^{-\varepsilon} dx \, \frac{1}{x} + \int_\varepsilon^b \frac{dx}{x}$$



$a < 0 \quad b > 0$

$$\cancel{\int_a^b dx \, \frac{1}{x}}$$

$$\int_a^b dx \; P\left(\frac{1}{x}\right) = \int_\varepsilon^b \frac{dx}{x} + \int_a^{-\varepsilon} \frac{dx}{x}$$

$$= \operatorname{Log}\left[\frac{b}{\varepsilon}\right] + \operatorname{Log}\left[\frac{\varepsilon}{|a|}\right]$$

$$= L_{a_3}\left[\frac{b}{|a|}\right]$$

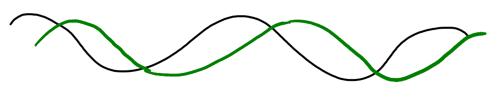
$$\lim_{\delta\to 0}\frac{1}{x+i\delta} = P\left(\frac{1}{x}\right) - i\pi\,\delta(x)$$

$$\mathrm{Re}\,\chi(\omega) = \frac{1}{Z}\sum_{n,m}\langle n|A|m\rangle\langle m|O|n\rangle P\left(\frac{e^{-\beta E_n} - e^{-\beta E_m}}{\omega + E_n - E_m}\right)$$

$$\mathrm{Im}\,\chi(\omega) = \frac{-\pi}{Z}\sum_{n,m}\langle n|A|m\rangle\langle m|O|n\rangle\left(e^{-\beta E_n} - e^{-\beta E_m}\right)\delta(\omega + E_n - E_m)$$

$$\langle A\rangle(\omega) = \chi(\omega)\,\lambda(\omega)$$

$$[\mathrm{Re}\chi + i\,\mathrm{Im}\chi]\quad e^{i\omega t}.$$



$$\mathrm{Im}\,\chi(\omega=0) = 0$$

$$\boxed{\mathrm{Re}\,\chi(\omega) = -\frac{1}{\pi}\int d\omega'\,P\frac{1}{\omega-\omega'}\,\mathrm{Im}\,\chi(\omega').}$$

Kramers–Kronig relations

[B. Yu Kuang Hu Am. J. Physics 57 821 (89)]

$$\chi(t) = \theta(t)\,f(t)$$

$$\chi(\omega) = \int\frac{d\omega'}{2\pi}\,\tilde\theta(\omega-\omega')\,\tilde f(\omega')$$

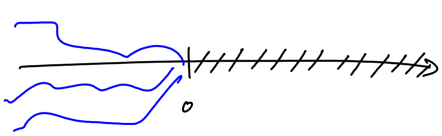
$$\tilde\theta(\omega) = \int_{-\infty}^{+\infty}dt\,e^{i\omega t}\,\theta(t) = \int_{0}^{+\infty}dt\,e^{i(\omega+i\delta)t}\,\theta(t)$$

$$= \frac{-1}{i(\omega+i\delta)} = \frac{i}{\omega+i\delta}.$$

$$\chi(\omega) = i\int d\omega'\,\frac{1}{\omega-\omega'+i\delta}\,\tilde f(\omega')$$

$$= i\int\frac{d\omega'}{2\pi}\,P\left(\frac{1}{\omega-\omega'}\right)\tilde f(\omega') + \frac{1}{2}\int d\omega'\,\delta(\omega-\omega')\,f(\omega')$$

$$\chi(\omega) = i\int\frac{d\omega'}{2\pi}\,P\left(\frac{1}{\omega-\omega'}\right)\tilde f(\omega') + \frac{1}{2}\tilde f(\omega)$$



$$f_1(t) = f_1(-t)$$

$$t>0 \quad f_1(t) = \chi(t).$$

$$f_2(t) = -f_2(-t)$$

$f_2(t) = -f_2(-t)$

$f_2(t>0) = \chi(t)$.

$$\tilde{f}(\omega) = \int_{-\infty}^{+\infty} dt\, e^{i\omega t} f(t) = \int_0^{+\infty} dt\, e^{i\omega t} f(t) + \int_{-\infty}^{0} dt\, e^{i\omega t} f(t).$$

$$= \underbrace{\int_0^{+\infty} dt\, e^{i\omega t} f(t)}_{\chi(\omega)} + \underbrace{\int_0^{+\infty} dt\, e^{-i\omega t} \underbrace{f(-t)}_{f(t)}}_{\chi(-\omega)}$$

Variation of energy.

$H_0 + H_{pert}(t)$.

$$H_{pert}(t) = \int dr\, \lambda(r,t)\, O_r + \lambda^*(r,t)\, O_r^+$$

$$= \lambda(t)\, O + \lambda^* O^+$$

$\lambda(t) = h e^{i\omega t}$. $\quad H_{pert}(t) = \left[h e^{i\omega t} O + h^* e^{-i\omega t} O^+\right]$

$$\langle H(t)\rangle = \mathrm{Tr}\left[\rho(t)\, H(t)\right]$$

$$\frac{\partial}{\partial t}\langle H(t)\rangle = \mathrm{Tr}\Big[\underbrace{\frac{\partial \rho}{\partial t}}_{-i[H(t),\rho(t)]} H(t)\Big] + \mathrm{Tr}\left[\rho(t)\frac{\partial H}{\partial t}\right]$$

$$\mathrm{Tr}\left[[H(t),\rho(t)]\, H(t)\right] = \mathrm{Tr}\left[\rho(t)\,[H(t),H(t)]\right] = 0$$

$$\frac{\partial}{\partial t}\langle H(t)\rangle = \left\langle \frac{\partial H}{\partial t}\right\rangle = h\, i\omega\, e^{i\omega t}\langle O\rangle_t - h^* i\omega\, e^{-i\omega t}\langle O^+\rangle_t$$

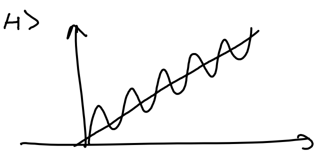
$$\langle O\rangle_t = \int dt'\, \chi_{OO}(t-t')\, h e^{i\omega t'} + \int dt'\, \chi_{OO^+}(t-t')\, h^* e^{-i\omega t'}$$

$$\langle O^+\rangle_t = \int dt'\, \chi_{O^+O}(t-t')\, h e^{i\omega t'} + \int dt'\, \chi_{O^+O^+}(t-t')\, h^* e^{-i\omega t'}$$

$$\langle O\rangle_t = e^{i\omega t}\int dt'\, \chi_{OO}(t-t')\, e^{i\omega(t'-t)} h + e^{-i\omega t}\int dt'\, \chi_{OO^+}(t-t')\, e^{+i\omega(t-t')} h^*$$

$$e^{i\omega t}\langle O\rangle_t = e^{i2\omega t}\,\underline{\quad} + \underline{\quad}$$

$\langle H\rangle$



Average energy injected in the system

$$\frac{\partial}{\partial t}\langle H\rangle = \left(\int dt'\ \chi_{OO^+}(t-t')\, h^* e^{i\omega(t-t')}\right) h\, i\omega$$
$$- \left(\int dt'\ \chi_{O^+O}(t-t')\, h\, e^{i\omega(t'-t)}\right) h^*\, i\omega$$

$$= i\omega\, hh^* \left[\int dt'\ \chi_{OO^+}(t-t')\, e^{i\omega(t-t')} - \int dt'\, \chi_{O^+O}(t-t')\, e^{i\omega(t'-t)}\right]$$

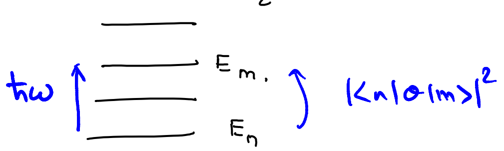
$$= i\omega\, hh^* \left[2i\ \mathrm{Im}\left[\int dt'\ \chi_{OO^+}(t-t')\, e^{i\omega(t-t')}\right]\right]$$

$$= -2\omega |h|^2\ \mathrm{Im}\left[\int dt'\ \chi_{OO^+}(t-t')\, e^{i\omega(t-t')}\right]$$

$$\frac{\partial}{\partial t}\langle H\rangle = -2\omega\ \mathrm{Im}\,\chi_{OO^+}(\omega)\ |h|^2$$

$$\mathrm{Im}\,\chi(\omega) = \frac{-\pi}{Z}\sum_{n,m}\langle n|A|m\rangle\langle m|O|n\rangle\left(e^{-\beta E_n} - e^{-\beta E_m}\right)\delta(\omega + E_n - E_m)$$

$$\mathrm{Im}\,\chi_{OO^+}(\omega) = -\frac{\pi}{Z}\sum_{n,m}|\langle n|O|m\rangle|^2\left(e^{-\beta E_n} - e^{-\beta E_m}\right)\delta(\omega + E_n - E_m)$$
$$= -\frac{\pi}{Z}\sum_{n,m}|\langle n|O|m\rangle|^2\left(1 - e^{-\beta\omega}\right)e^{-\beta E_n}\,\delta(\omega + E_n - E_m)$$
$$= -\frac{\pi}{Z}\left(1 - e^{-\beta\omega}\right)\sum_{n,m}|\langle n|O|m\rangle|^2\, e^{-\beta E_n}\,\delta(\omega + E_n - E_m).$$



Connection with imaginary time

$$\chi(\tau) = -\left\langle T_\tau\ \hat{A}(\tau)\ O(0)\right\rangle \qquad \tau > 0$$

$$\chi(\tau) = -\frac{1}{Z}\sum_n \langle n|\, e^{-\beta H}\, e^{H\tau} A\, e^{-H\tau}\, O\, |n\rangle$$

$|m\rangle\langle m|$

$$= -\frac{1}{Z}\sum_{n,m}\langle n|\, e^{-\beta H}\, e^{H\tau} A\, e^{-H\tau}|m\rangle\langle m|\, O\,|n\rangle$$
$$= -\frac{1}{Z}\sum_{n,m} e^{-\beta E_n}\, e^{\tau(E_n - E_m)}\ \langle n|A|m\rangle\langle m|O|n\rangle$$

$$\chi(i\omega_p) = \int_0^\beta d\tau\ e^{i\omega_p\tau}\ \chi(\tau) \qquad \omega_p = \frac{2\pi}{\beta} n$$

$$\chi(i\omega_p) = \int_0 d\tau \, e \quad \chi(\tau) \qquad \omega_p = \frac{}{\beta} p.$$

$$= -\frac{1}{Z} \sum_{n,m} e^{-\beta E_n} \int_0^\beta d\tau \, e^{i\omega_p \tau} e^{\tau(E_n - E_m)} \quad \langle n|A|m\rangle \langle m|O|n\rangle$$

$$-\frac{1}{Z} \sum_{n,m} e^{-\beta E_n} \frac{e^{i\omega_p \beta + \beta(E_n - E_m)} - 1}{i\omega_p + (E_n - E_m)} \quad \langle n|A|m\rangle \langle m|O|n\rangle$$

$$= -\frac{1}{Z} \sum_{n,m} e^{-\beta E_n} . \frac{e^{\beta(E_n - E_m)} - 1}{i\omega_p + E_n - E_m} \quad \langle n|A|m\rangle \langle m|O|n\rangle$$

$$= \frac{1}{Z} \sum_{n,m} \langle n|A|m\rangle \langle m|O|n\rangle \frac{e^{-\beta E_n} - e^{-\beta E_m}}{i\omega_p + E_n - E_m.} = \chi_{\substack{\text{time} \\ \text{ordered}}}(i\omega_p)$$

$$\chi_{ref}(\omega) = \frac{1}{Z} \sum_{n,m.} \langle n|A|m\rangle \langle m|O|n\rangle \frac{e^{-\beta E_n} - e^{-\beta E_m}}{\omega + E_n - E_m + i\delta}$$

Response A to a perturbation O

$$\chi(\tau) = -\langle T_\tau \, A(\tau) \, O(0) \rangle \qquad \Rightarrow \text{path integral}$$

$$\chi(i\omega_p) = \int_0^\beta d\tau \, e^{i\omega_p \tau} \, \chi(\tau) \qquad \omega_p = \frac{2\pi}{\beta} p.$$

Analytic. continuation.

$$\chi(i\omega_p \to \omega + i\delta) \quad \to \text{physical retarded correlation function.}$$

$$-i\,\theta(t) \langle [\hat{A}(t), \hat{O}(0)] \rangle$$

Summary

Quantum system H

$$Z = \mathrm{Tr}[e^{-\beta H}] \quad \to \quad \int \mathcal{D}\phi \; e^{-\int_0^\beta d\tau \, \mathcal{H}[\tau]}$$

$$\chi = -\langle T_\tau \, O_1(\tau_1) \, O_2(\tau_2) \cdots \rangle \qquad O(\tau) = e^{H\tau} O e^{-H\tau}$$

$$\hookrightarrow \frac{1}{Z} \int \mathcal{D}\phi \; O(\phi(\tau_1)) \, O(\phi(\tau_2)) \ldots \; e^{-\int_0^\beta d\tau \, \mathcal{H}(\tau)}$$

- $\chi(\tau) = -\langle T_\tau \, A(\tau) \, O(0) \rangle \to \chi(i\omega_n)$

  $\omega_n = \frac{2\pi}{\beta} n.$

- $\chi_{ret}(t) = -i\,\theta(t) \langle [A(t), O(0)] \rangle \to \chi_{ret}(\omega)$

$$\chi(i\omega_n \to \omega + i\delta) = \chi_{ret}(\omega) \Leftarrow \text{physical object}$$

[linear response]