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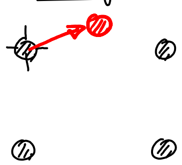
Examples

I] Quantum crystal:

1) Definition.



R_i° : equilibrium position.

$V(R_i - R_j)$: interaction.

$\vec{R}_i = \vec{R}_i^\circ + \vec{u}_i$

u_i : operator

$u_i \left| \begin{matrix} x_i \\ y_i \\ z_i \end{matrix} \right.$ -- Measuring the position of the particle.

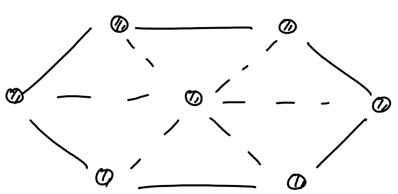
$$H = \sum_{i=1}^{N} \frac{\hat{P}_i^2}{2m} + \frac{1}{2}\sum_{i,j} V(\hat{R}_i - \hat{R}_j)$$

Physical realizations:

→ ions in a crystalline lattice (→ phonons)

→ Crystal of electrons.

E. Wigner if the density is low → electrons crystallize



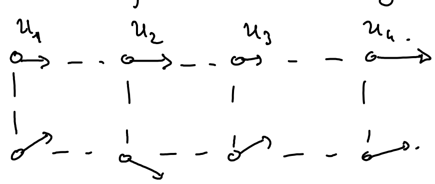
Wigner crystal.

$n \sim 10^{11}\ e/cm^2$

$$V(R_i - R_j) = V(R_i^\circ - R_j^\circ + u_i - u_j)$$
$$= V(R_i^\circ - R_j^\circ) + A_{ij}\left[u_i - u_j\right]^2$$

$$H = \sum_{i=1}^{N} \frac{P_i^2}{2m} + \frac{1}{2}\sum_{i,j} A_{ij}\left[u_i - u_j\right]^2$$

A_{ij} only exist for nearest neighbors. → Continuum limit



$$u_i \to u(r) \qquad [P_i, u_i] = i\hbar$$
$$P_i \to \Pi(r) \quad \downarrow \quad [\Pi(r), u(r')] = i\hbar\, \delta(r-r')$$

$$H = \int dr \left[\frac{\Pi^2(r)}{2m\rho_0} + \frac{1}{2}\rho_0 c (\nabla u(r))^2 \right]$$

$$Z = \mathrm{Tr}\left[e^{-\beta H}\right] = \mathrm{Tr}\left[e^{-\frac{\beta}{N}H} e^{-\frac{\beta}{N}H} \cdots \right]$$

$$Z = \int \mathcal{D}u(r,\tau)\, \mathcal{D}\Pi(r,\tau)\; e^{\int_0^\beta d\tau \left[\int i\Pi \partial_\tau u\, dr - H[\Pi, u]\right]}$$

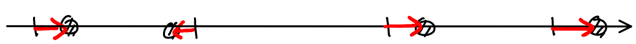
$$= \int \mathcal{D}u(r,\tau)\, \mathcal{D}\Pi(r,\tau)\; e^{\int_0^\beta d\tau \int dr \left[i\Pi(r,\tau)\partial_\tau u(r,\tau) - \frac{\Pi^2}{2m\rho_0} - \frac{1}{2}\rho_0 c (\nabla u)^2 \right]}$$

$$-\frac{\Pi^2}{2m\rho_0} + i\Pi \partial_\tau u = -\frac{[\Pi - i\partial_\tau u\, m\rho_0]^2}{2m\rho_0} - \frac{1}{2} m\rho_0 (\partial_\tau u)^2$$

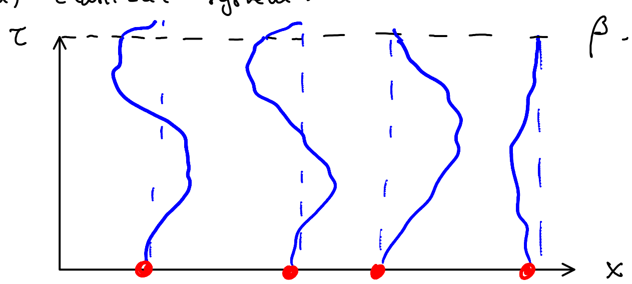
$$Z = \int \mathcal{D}u(r,\tau)\; e^{-\int_0^\beta d\tau \int dr \left[\frac{1}{2} m\rho_0 (\partial_\tau u)^2 + \frac{1}{2}\rho_0 c (\nabla u)^2\right]}$$

$$Z = \int \mathcal{D}u(r,\tau)\; e^{-\frac{1}{2}\int_0^\beta d\tau \int d^d r\, \rho_0 \left[m(\partial_\tau u)^2 + c(\nabla u)^2 \right]}$$

$\underline{d=1}$



Equivalent classical system.



2D classical "crystal" of lines.

even at $T=0$ Quantum fluctuations melt the crystal.

$\underline{d=2}$ $\to$ $d=3$ classical crystal. $(T=0)$

↳ possibility of perfect order

$T \neq 0$ $\to$ Finite size for the classical problem $\to d=2$

$\hookrightarrow$ Melting of the crystal.

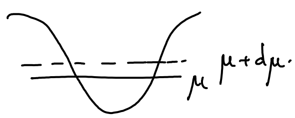
$$\hat{\rho}(x) = \rho_0 - \rho_0 \nabla \hat{u}(r) + \rho_0 \sum_K e^{iK[r - \hat{u}(r)]}$$

$$\langle T_\tau \, \hat{\rho}(x_1,\tau) \, \hat{\rho}(x_2,0) \rangle \longrightarrow \frac{\int \mathcal{D}u \; e^{-S} \; \rho(u(x_1,\tau)) \, \rho(u(x_2,0))}{\int \mathcal{D}u \; e^{-S}}$$

Compressibility

$$\kappa = \frac{-1}{\Omega} \frac{\partial \Omega}{\partial P} \qquad F = \Omega f\left(\frac{N}{\Omega}\right)$$

$$\kappa = \frac{\partial N}{\partial \mu}$$



dN

$$H - \mu \int dr \, \rho(r) = H_\mu$$

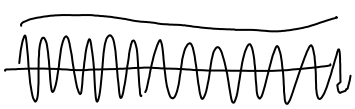
$$\left\langle \int dr_0 \, \rho(r_0) \right\rangle = \int dr \; \frac{1}{Z_\mu} \int \mathcal{D}u \; \rho(u(r_0,\tau_0)) \; e^{-S_\mu}$$

$$Z_\mu = \int \mathcal{D}u \; e^{-S_\mu}$$

$$S_\mu = \frac{1}{2} \int_0^\beta d\tau \int d^d r \; \rho_0 \left[m(\partial_\tau u)^2 + c(\nabla u)^2 \right] - \mu \int dr \, d\tau \; \rho(u(r,\tau))$$

$$\int dr \, \delta\rho(u(r,\tau)) = -\rho_0 \int dr \; \nabla_r u(r,\tau) + \rho_0 \int dr \sum_K e^{iK[r - u(r,\tau)]}$$

K : vectors of the reciprocal lattice $\quad K \simeq \frac{2\pi}{a} \quad$ a : lattice spacing



$$\int dr \, \delta\rho(u(r,\tau)) \approx -\rho_0 \int dr \; \nabla_r u(r,\tau).$$

$$S_\mu = \frac{1}{2} \int_0^\beta d\tau \int d^d r \; \rho_0 \left[m(\partial_\tau u)^2 + c(\nabla u)^2 \right] + \mu \rho_0 \int dr \, d\tau \; \nabla_r u(r,\tau)$$

$$\rho_0 \left\langle \int dr_0 \; \nabla u(r_0,\tau_0) \right\rangle = \rho_0 \int dr_0 \frac{\int \mathcal{D}u \; \nabla u(r_0,\tau_0) \; e^{-S^0 + \mu \rho_0 \int dr \, d\tau \, \nabla_r u(r,\tau)}}{\int \mathcal{D}u \; e^{-S^0 + \mu \rho_0 \int dr \, d\tau \, \nabla_r u(r,\tau)}}$$

τ^0

$$= \langle \quad \rangle_{S_0} + \mu \rho_0^2 \int dr_0 \int dr\,d\tau \frac{\int \mathcal{D}u \; \nabla u(r_0,\tau_0)\, \nabla u(r,\tau) \; e^{-S^0}}{\int \mathcal{D}u \; e^{-S_0}}$$

$$= \langle \quad \rangle_{S_0} + \mu \rho_0^2 \int dr_0 \int dr\,d\tau \; \langle \nabla u(r_0,\tau_0)\, \nabla u(r,\tau) \rangle_{S_0}$$

$$u(q,\omega_n) = \int_0^\beta d\tau \int dr \; e^{i(\omega_n \tau - q r)} \; u(r,\tau)$$

$$u(r,\tau) = \frac{1}{\beta\Omega} \sum_{\omega_n q} e^{i(qr - \omega_n \tau)} \; u(q,\omega_n)$$

$$S^0 = \frac{\rho_0}{2} \int dr \int_0^\beta d\tau \; \frac{1}{(\beta\Omega)^2} \sum_{\substack{\omega_1 q_1 \\ \omega_2 q_2}} \Big[m\,(i\omega_1)(i\omega_2)\, u(q_1\omega_1)\, u(q_2\omega_2) \; e^{i(q_1 r - \omega_1\tau)} e^{i(q_2 r - \omega_2\tau)}$$

$$+ \; c\,(iq_1)(iq_2)\, u(q_1\omega_1)\, u(q_2\omega_2) \Big]$$

$$= \frac{\rho_0}{2} \frac{1}{\beta\Omega} \sum_{q,\omega_n} \left[m\,\omega^2 + c\,q^2 \right] u^*(q\omega)\, u(q\omega)$$

$$\int dr_0 \int dr \int d\tau \quad \langle \nabla u(r_0,\tau_0) \; \nabla u(r,\tau) \rangle$$

$$\frac{1}{(\beta\Omega)^2} \sum_{\substack{q_1\omega_1 \\ q_2\omega_2}} (iq_1)(iq_2) \; e^{i(q_1 r_0 - \omega_1 \tau_0)} \, e^{i(q_2 r - \omega_2 \tau)}$$

$$\langle u(q_1\omega_1)\, u(q_2\omega_2) \rangle$$

$$= \int dr_0 \int dr \int d\tau \; \frac{1}{(\beta\Omega)^2} \sum_{\substack{q_1\omega_1 \\ q_2\omega_2}} (iq_1)(iq_2) \; e^{i(\,)} e^{i(\,)} \; \frac{\beta\Omega}{\rho_0} \, \delta_{q_1,-q_2} \, \delta_{\omega_1,-\omega_2} \, \frac{1}{m\omega_1^2 + c q^2}$$

$$\frac{1}{\rho_0} \int dr_0 \int dr \int d\tau . \; \frac{1}{\beta\Omega} \sum_{q,\omega} e^{i(q_1 r_0 - \omega_1 \tau_0)} \, e^{-i(q_1 r - \omega_1 \tau)} \; \frac{q^2}{m\omega_1^2 + c q^2}$$

$$\omega_1 = 0$$

$$\iint \langle \nabla u \; \nabla u \rangle = \lim_{q_1 \to 0} \; \omega_1 = 0 \quad \frac{q^2}{m\omega_1^2 + c q^2}$$

$$= \lim_{q_1 \to 0} \frac{q^2}{c\,q^2} = \frac{1}{c}$$

$$K = \frac{\rho_0}{c} \qquad \text{Compressibility is constant}$$

Linear response theory.

$$H - \int dr \; \mu(r,t) \, [\rho(r) - \rho_0]$$

$$\langle [\rho(r_0, t_0) - \rho_0] \rangle \rightarrow \int dr dt' \; \chi(r_0 - r, t_0 - t) \; \mu(r,t)$$

$$\chi_{ret}(r_0 - r, t_0 - t) = -i \, \theta(t_0 - t) \langle [\delta\hat{\rho}(r_0, t_0), \delta\hat{\rho}(r,t)] \rangle$$

Compute: $\quad - \langle T_\tau \; \delta\hat{\rho}(r_0, \tau_0) \; \delta\hat{\rho}(r, \tau) \rangle = \chi(r_0 - r, \tau_0 - \tau)$

$\hookrightarrow \quad \chi(\omega_n) = \int_0^\beta d\tau \; \chi(r, \tau)$

$$\langle T_\tau \; \nabla\hat{u}(r_0, \tau_0) \; \nabla\hat{u}(r, \tau) \rangle$$

$$= \frac{1}{Z_0} \int \mathcal{D}u \cdot \; \nabla u(r_0, \tau_0) \; \nabla u(r, \tau) \; e^{-\frac{\rho_0}{2}\int_0^\beta d\tau \int dr \, [m \dot{u}^2 + c(\nabla u)^2]}$$

$$\langle u^*(q, \omega_n) \; u(q, \omega_n) \rangle = \frac{\Omega\beta}{\rho_0} \; \frac{1}{m\omega_n^2 + cq^2}.$$

$$\int dr_0 \cdot \; \langle [\delta\hat{\rho}(r_0, t) \; \delta\hat{\rho}(0, 0)] \rangle \; e^{iqr_0}.$$

$$= \langle [\delta\hat{\rho}(q, t) \; \delta\hat{\rho}(-q, 0)] \rangle$$

$$\delta\rho(r,t) = \nabla u(r,t) \qquad \Rightarrow \qquad \delta\rho(q) = iq \; u(q).$$

$$\langle \delta\rho(q, \omega) \rangle = \chi_{ret}(q, \omega) \; \mu(q, \omega)$$

$$\chi_{ret}(q, \omega) = -i \int_0^{+\infty} dt \; e^{i(\omega + i\delta)t} \; q^2 \; \langle [u^*(q,t), u(q,0)] \rangle$$

$$\chi_{\text{time ordered}}(q, \omega_n) = \frac{q^2}{m\omega_n^2 + cq^2} \qquad \begin{aligned} & m\omega_n^2 \\ & = -m(i\omega_n)^2. \end{aligned}$$

$\downarrow$

$$\chi_{ret}(q, \omega) = \frac{q^2}{-m[\omega + i\delta]^2 + cq^2}$$

$$\chi_{ret}(q,\omega) = \frac{1}{-m[\omega + i\delta]^2 + cq^2}$$

$$= \frac{q^2}{-m\omega^2 + cq^2 - 2im\omega\delta}$$

<u>Compressibility:</u>

μ: independant of space and time. $\omega \to 0$ $q \to 0$

$$\frac{\partial N}{\partial \mu} \longrightarrow \frac{\langle \delta\rho \rangle}{\mu} = \chi(q=0, \omega=0)$$

$\omega \to 0$ first $\qquad \chi = \frac{1}{c}$

$q \to 0$ first $\qquad \chi = 0$

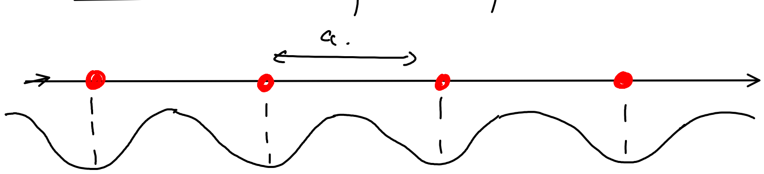
\# <u>thermodynamic quantity</u>: one <u>must</u> take $\omega \to 0$ first to get a time independant potential. then spatial limits

<u>Opposite order:</u>

↳ transport properties. $\qquad \frac{\partial \rho}{\partial t} + \nabla . j = 0$

Imaginary time: $\qquad \int_0^\beta d\tau \; u(r,\tau) = u(r, \omega_n = 0)$

4) <u>Crystal in a periodic potential.</u>



$$H = \sum_i \frac{P_i^2}{2m} + \frac{c}{2}\sum_i (u_{i+1} - u_i)^2 + \int dx\, \rho(x)\, V(x)$$

$$V(x) = V_0 \cos(Q x) \qquad Q = \frac{2\pi}{a}.$$

$$H = \int dx \left[\frac{\Pi^2}{2m} + \frac{1}{2}\rho_0 c (\nabla u)^2 \right] + H_v$$

$$H_v = V_0 \int dx \; \cos(Qx) \left[\rho_0 - \rho_0 \nabla \hat{u} + \rho_0 \sum_K e^{iK[x - \hat{u}(x)]} \right]$$

$$S = \frac{\rho_0}{2} \int_0^\beta d\tau \, dx \left[m (\partial_\tau u)^2 + c (\partial_x u)^2 \right] + \int_0^\beta d\tau \int dx$$

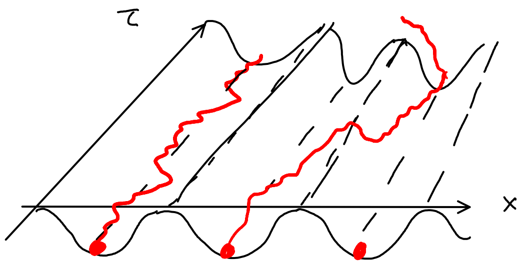
$$\Big[\underbrace{\cos}_{\frac{e^{iQx}+e^{-iQx}}{2}}(\underset{\uparrow \, \frac{2\pi}{a}}{Qx}) \Big[\rho_0 - \rho_0 \nabla u(x,\tau) + \rho_0 \sum_K e^{\underset{\uparrow \, \frac{2\pi}{a} n.}{i K}[x - u(x,\tau)]} \Big] \Big]$$

$e^{iQx} \quad e^{-iKx} \quad \to$ non oscillating term

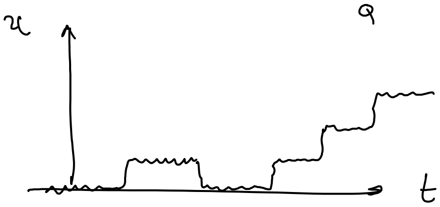
$$S_v = \int_0^\beta d\tau \int dx \; \rho_0 V_0 \cos(Q u(x,\tau))$$

$$S = \frac{\rho_0}{2} \int_0^\beta d\tau \, dx \left[m \underset{\downarrow \, \pi^2.}{(\partial_\tau u)^2} + c (\partial_x u)^2 \right] - V_0 \rho_0 \int d\tau \, dx \cos(Qu)$$

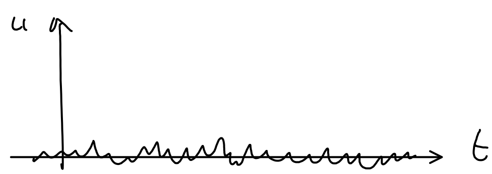
Sine-Gordon Hamiltonian. $\qquad B(r,\tau) = \langle [u(r,\tau) - u(0,0)]^2 \rangle$



V_0 small. $\qquad u \approx 0 \quad \frac{2\pi}{Q} \quad \frac{4\pi}{Q} \quad \frac{6\pi}{Q}$



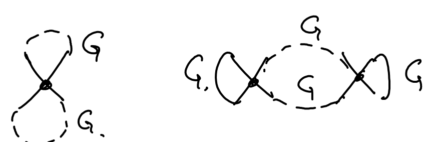
V_0 large



$\varphi^\dagger G^{-1} \varphi + \varphi^4$

$\cos(\varphi)$

$\varphi^2 + \varphi^4 + \varphi^6 + \varphi^8 + \cdots$



$$\varphi^2 + \varphi^4 + \varphi^6 + \varphi^8 + \cdots$$

<u>Connection with ....</u> :

$$Z = \int \mathcal{D}u \; e^{-S_0 - S_V} = Z_0 \underbrace{\frac{\int \mathcal{D}u \; e^{-S_0 - S_V}}{\int \mathcal{D}u \; e^{-S_0}}}_{\langle e^{-S_V'} \rangle_{S_0}}$$

$$\left\langle e^{-\int_{-\beta/2}^{\beta/2} d\tau \int dx \frac{y}{V_0 \rho_0} \cos(Q u(x,\tau))} \right\rangle$$

$$= \langle 1 - y \int d\tau dx \; \cos(Q u(x,\tau)) + \frac{1}{2} y^2 \int dx_1 d\tau_1 \int dx_2 d\tau_2 \cos(Q u_1) \cos(Q u_2)$$

$$+ \cdots$$

$$S_0 = \frac{\rho_0}{\Omega\beta} \sum_{q,\omega_n} \left[m\omega_n^2 + c q^2 \right] u^*_{q\omega_n} u_{q\omega_n}.$$

$$\langle \cos(Qu) \rangle \Rightarrow \langle e^{i \pm Q u} \rangle$$

$$\frac{1}{Z_0} \int \mathcal{D}u \; e^{-\frac{\rho_0}{\Omega\beta} \sum_{q\omega_n} [m\omega_n^2 + cq^2] u^*_{q\omega_n} u_{q\omega_n}} \; e^{\pm i Q \frac{1}{\Omega\beta} \sum_{q\omega_n} e^{i(qr - \omega\tau)} u_{q\omega}}$$

$$e^{iQu} = e^{iQ \frac{1}{2} \left[\sum_{q\omega} e^{i\vec{q}\vec{r}} u_{\vec{q}} + e^{-i\vec{q}\vec{r}} u^*_{\vec{q}} \right]} \qquad \vec{q} = (q, \omega)$$

$$e^{-\frac{1}{\Omega\beta} \sum_{q\omega_n} [m\omega_n^2 + cq^2] \rho_0 \left[u^* + iQ e^{i\vec{q}\vec{r}} \frac{1}{\rho_0 [m\omega_n^2 + cq^2]} \right] \left[u + iQ e^{-iqr} \frac{1}{} \right]}$$

$$e^{-\frac{Q^2}{2\Omega\beta} \sum_{\omega_n q} \frac{1}{\rho_0 [\omega_n^2 m + cq^2]}}$$

$$\langle e^{iQu(r,\tau)} \rangle_{S_0} = e^{-\frac{1}{2} \frac{1}{\Omega\beta} \sum_{q\omega_n} \frac{1}{\rho_0 [\omega_n^2 m + cq^2]}} = e^{\frac{1}{2} \langle (iQu)^2 \rangle}$$

$$\omega_n = \frac{2\pi}{\beta} n. \qquad \beta \to \infty \qquad \frac{1}{\beta} \sum_{\omega_n} = \frac{1}{(2\pi)} \int d\omega$$

$$\frac{1}{\Omega} \sum_{k} \qquad \frac{1}{2\pi} \int dk. \qquad k = \frac{2\pi}{\Omega} p.$$

$$e^{-\frac{1}{2} \frac{1}{(2\pi)^2} \int^{\Lambda} dq \, d\omega \frac{1}{\rho_0 [m\omega^2 + cq^2]}} = 0$$

Λ ... cutoff : lattice spacing

$$\int_0^{\Lambda} d^2k \, \frac{1}{k^2} = \infty \sim \mathrm{Log}\left[\frac{\Lambda}{k_{min}}\right]$$

- Cutoff : lattice spacing.

$k_{min} \approx T, 1/L \ldots$

$\langle \cos(Q u) \rangle$ $\qquad S_0 = \int (\partial_\tau u)^2 + (\partial_x u)^2.$

$u \qquad u + \frac{\pi}{Q}.$ $\qquad$ weight of $u(r,\tau)$ $\quad u(r,\tau) + \frac{\pi}{Q}.$ are the same.

$$\langle \cos(Q u(r_1,\tau_1)) \cos(Q u(r_2,\tau_2)) \cos(Q u(r_3,\tau_3)) \rangle$$

All the odd terms go away.

Second order :

$$\langle \cos(Q u(r_1,\tau_1)) \cos(Q u(r_2,\tau_2)) \rangle$$

$$\langle e^{i\varepsilon_1 Q u(r_1\tau_1)} \, e^{i\varepsilon_2 Q u(r_2\tau_2)} \rangle \qquad \varepsilon_1 = \pm 1 \qquad \varepsilon_2 = \pm 1.$$

$$\langle e^{i\varepsilon_1 Q u(r_1,\tau_1)} \, e^{i\varepsilon_2 Q u(r_2,\tau_2)} \, e^{i\varepsilon_3 \cdot} \quad e^{i\varepsilon_4} \cdots \rangle$$

$$\langle e^{i\sum_{j=1}^{2p} \varepsilon_j \cdot Q \, u(x_j,\tau_j)} \rangle = \langle e^{i\sum_{j=1}^{2p} \varepsilon_j Q \frac{1}{\beta\Omega}\sum_{q\omega_n} u_{q\omega_n}[e^{i\vec{q}\vec{r}_j}]} \rangle$$

$$\langle e^{\frac{1}{2\beta\Omega}\sum_{q\omega_n} u_{q\omega_n}\left(iQ\sum_{j=1}^{2p} \varepsilon_j e^{i\vec{q}\vec{r}_j}\right) + u^*_{q\omega_n}(\quad)^*} \rangle$$

$$e^{-\frac{1}{2}\frac{Q^2}{\beta\Omega}\sum_{q\omega_n}\frac{1}{\rho_0[m\omega^2 + cq^2]}\left(\sum_{j=1}^{2p} \varepsilon_j e^{i\vec{q}\vec{r}_j}\right)\left(\sum_{j'=1}^{2p} \varepsilon_{j'} e^{-i\vec{q}\vec{r}_{j'}}\right)}$$

2 cosines. $\qquad (\varepsilon_1 e^{iqr_1} + \varepsilon_2 e^{iqr_2})(\varepsilon_1 e^{-iqr_1} + \varepsilon_2 e^{-iqr_2})$

$$1 + \varepsilon_1\varepsilon_2 \, e^{iq(r_1-r_2)} + e^{-iq(r_1-r_2)} + 1$$

$$= 2 + 2\varepsilon_1\varepsilon_2 \cos(q(r_1-r_2))$$

$$e^{-\frac{Q^2}{2\beta\Omega}\sum_{q\omega_n}\frac{1}{\rho_0[m\omega^2+cq^2]}\left[2 + 2\varepsilon_1\varepsilon_2\cos(q(r_1-r_2))\right]}$$

$$e^{-\frac{Q^2}{2}\frac{1}{(2\pi)^2}\int^{\Lambda} d^2k \, \frac{1}{k^2}\left[1 + \varepsilon_1\varepsilon_2\cos(\vec{k}\cdot\vec{r})\right]} \qquad \vec{r} = \vec{r}_1 - \vec{r}_2$$

$$\langle e^{iQu} \, e^{iQu} \rangle \qquad \langle e^{-iQu} \, e^{-iQu} \rangle \longrightarrow 0$$

$$\langle e^{iQu} \, e^{-iQu} \rangle \qquad \langle e^{-iQu} \, e^{iQu} \rangle$$

$$\langle e^{iQu}\; e^{-iQu} \rangle \qquad \langle e^{-iQu}\; e^{iQu} \rangle$$

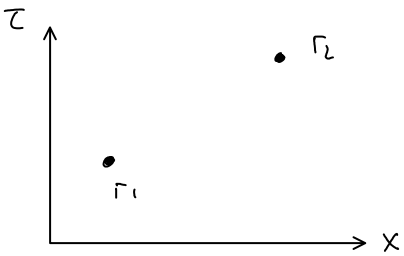
$$\int d^2k \; \frac{1}{k^2} \left[1 - \cos(kr) \right] = \int_0^{\Lambda} \frac{dk}{k} \left[1 - \cos(kr) \right] \approx \int_{1/r}^{\Lambda} \frac{dk}{k}$$

$$= \operatorname{Log}\left[\Lambda r \right] \approx \operatorname{Log}\left[\frac{r}{a} \right] \qquad r = r_1 - r_2 = \sqrt{(x_1-x_2)^2 + (\tau_1-\tau_2)^2}$$

$$\int \underbrace{dx_1\, d\tau_1}_{d^2 r_1} \; \underbrace{dx_2\, d\tau_2}_{d^2 r_2} \; \langle \underbrace{\cos(Qu_1)\; \cos(Qu_2)}_{(e^{iQu}+e^{-iQu})\,(e^{iQu}+e^{-iQu})} \rangle \qquad e^{-Q^2 c \operatorname{Log}\left[\frac{r_1-r_2}{a}\right]}$$

particle charge $-Q$.

particle charge $+Q$



have interactions $Q^2 \operatorname{Log}\left[\frac{r_1-r_2}{a}\right]$

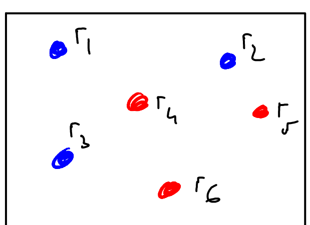
Coulomb interaction in 2 dimensions

$$\Delta V(x) = Q\, \delta(x)$$

$$V(q) = \frac{1}{q^2} \longrightarrow V_{2D}(r) = \operatorname{Log}[r]$$

Coulomb gaz!

Set of N particles / retain the neutral configurations



$$Z_N = \int dr_1 dr_2 \ldots dr_N \; e^{-\beta \sum_{i,j} Q_i Q_j \operatorname{Log}[r_i - r_j]}$$

Work with a fixed chemical potential.

$$Z = \sum_{N=0}^{+\infty} e^{-\beta[H - \mu N]} = \sum_{N=0}^{+\infty} \left(e^{+\beta\mu} \right)^N \int dr \; e^{-\beta H_{N\,\text{particles}}}$$

$$= \sum_{N=0}^{+\infty} \left(e^{\beta\mu} \right)^N Z_{N\,\text{particles}}$$

Quantum Problem.	Classical Problem.
Quantum crystal + periodic potential.	Coulomb gaz
$d=1$ $\quad T=0$	2 dim problem.
m, c, Q	Q charge of the particles
V_0: strength of periodic pot.	$V_0 = e^{\beta\mu}$: fugacity of the particl

Quantum Problem.	Classical Problem.
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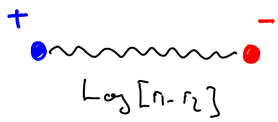
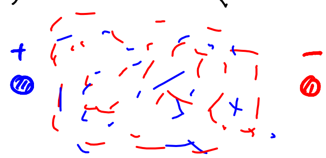
m, c, Q	Q charge of the particles
V_0: strength of periodic pot.	$V_0 \equiv e^{\beta\mu}$: fugacity of the particles

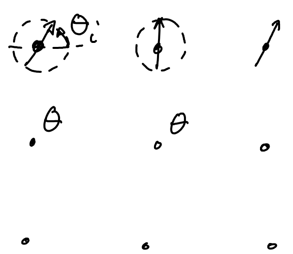
$$Z/Z_0 = \sum_{p=0}^{+\infty} y^{2p} \int d^2r_1\, d^2r_2 \ldots d^2r_{2p}\; e^{-\frac{1}{2}\sum_{i,j} Q_i Q_j \operatorname{Log}\left[\frac{r_i - r_j}{a}\right]}$$

$$B(r) = \langle [u(x,\tau) - u(0,0)]^2 \rangle \qquad \langle e^{i u(x,\tau)}\, e^{-i u(0,0)} \rangle = e^{-\frac{1}{2} B(r)}.$$





X-Y Model:



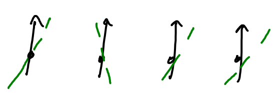
$$H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j$$

$$\vec{S}_i = [\cos\theta_i, \sin\theta_i]$$

$$H = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j).$$

$$Z = \int_{-\pi}^{\pi} d\theta_1 \ldots d\theta_N \quad e^{+\beta J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)}$$

$$\langle S_r^{(x)} S_0^{(x)} \rangle = \langle \cos(\theta_r) \cos(\theta_0) \rangle$$



$\theta_{i+1} - \theta_i$ small. $\to$ expand. $\cos(\theta_i - \theta_j)$

$$= 1 - \frac{1}{2} [\theta_i - \theta_j]^2.$$

$$\to \int dx\,dy \ \frac{1}{2}\left[(\partial_x \theta)^2 + (\partial_y \theta)^2 \right]$$

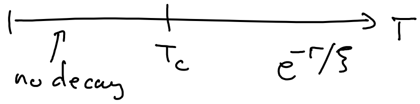
$$e^{-\frac{\beta J}{2} \int dx\,dy\, \left[(\partial_x \theta)^2 + (\partial_y \theta)^2 \right]}$$

$$\langle S \rangle = e^{\langle \cos\theta \rangle} = 0$$

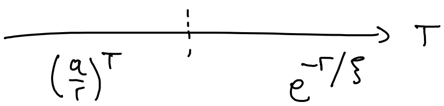
$$\langle \sin\theta \rangle = 0$$

$$\langle \cos\theta(r)\cos\theta(0)\rangle = e^{-c_1 T \operatorname{Log}[r/a]} = \left(\frac{a}{r}\right)^{Tc_1}$$

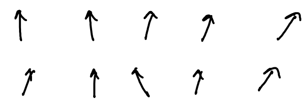
One has to have $\langle S^r S^0 \rangle \sim e^{-r/\xi}$ at "high" temperature

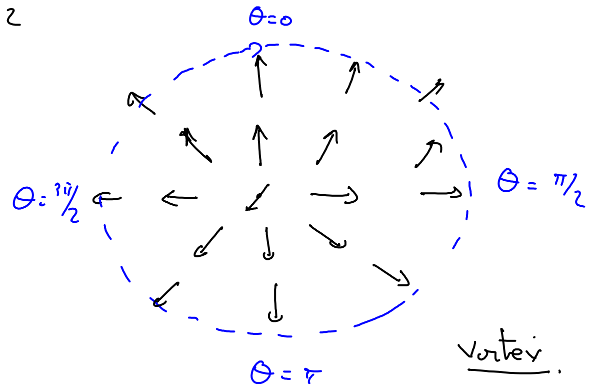


$d=3$



$d=2$





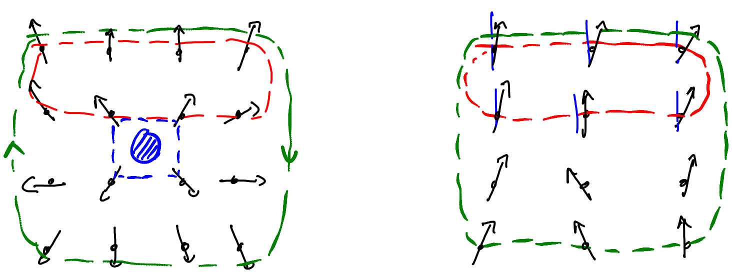
Vortex.

$$\int d\ell \cdot \nabla\theta(\ell) = 2\pi.$$

$\nabla\theta$ finite.

$\cos(\theta_i - \theta_j)$ ↑ ~~↓~~

$(\nabla\theta)^2$ ↑ ↑ smooth deviations between neighbors.



$$\oint d\vec{\ell} \cdot \vec{\nabla}\theta = 0 \qquad \oint d\vec{\ell} \cdot \vec{\nabla}\theta = 0$$

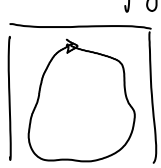
$$\oint d\vec{\ell} \cdot \vec{\nabla}\theta = 2\pi \qquad \oint d\vec{\ell} \cdot \vec{\nabla}\theta = 0$$

$$\int d\ell \cdot \vec{A} = \int d\vec{S} \operatorname{rot} \vec{A}$$

Conf for which $\nabla\theta$ is well defined,

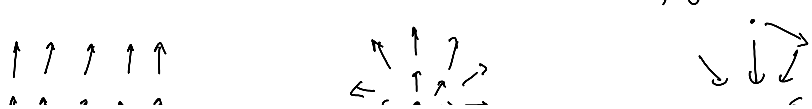
$$\operatorname{rot} \cdot \nabla \cdot \theta = 0$$

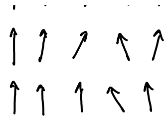
Class the configurations by their topological number



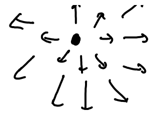
$$\int_C d\theta = 2\pi n \qquad n : \text{integer}.$$

n vorticity of the configuration.

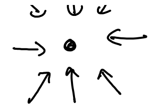




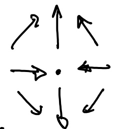
Spin waves.



Vortex (vorticity +1)

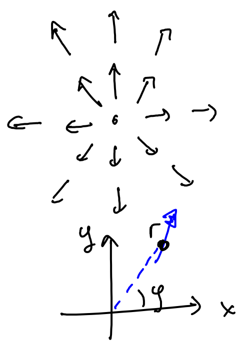


Vortex (vorticity +1)



Vortex (vorticity −1)

Energy of a vortex



$$E = -J\sum_{\langle ij\rangle} \cos(\theta_i - \theta_j) \;\to\; \int_{a^2}^{\Omega} \frac{J}{2} (\nabla\theta)^2 \, d^2r$$

$$\theta(x,y) = \varphi \qquad \begin{array}{l} x = r\cos\varphi \\ y = r\sin\varphi \end{array}$$

$$\vec{\nabla} = \left(\partial_r \;,\; \frac{1}{r}\partial_\varphi \right)$$

$$\nabla\theta = \left(0 \,,\, \frac{1}{r}\right)$$

$$E = \frac{J}{2} \int_a^L r\,dr \int_0^{2\pi} d\varphi \, \frac{1}{r^2} = J\pi \int_a^L \frac{dr}{r} = J\pi \operatorname{Log}\left[\frac{L}{a}\right]$$

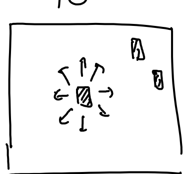
$$E_{\text{vortex}} = J\pi \operatorname{Log}\left[\frac{L}{a}\right]$$

Berezinskii V.L. Sov. Phys. JETP 32 493 (71)

Kosterlitz JM + Thouless DJ J. Phys. C 6 1181 (73)

Kosterlitz JM J. Phys. C 7 1046 (74)

Entropy of a vortex:



a^2

Configurations

$$N = \left(\frac{L}{a}\right)^2$$

$$S = \operatorname{Log} N \equiv 2\operatorname{Log}\left(\frac{L}{a}\right)$$

$$F_{\text{vortices}} = J\pi \operatorname{Log}\left[\frac{L}{a}\right] - 2T \operatorname{Log}\left(\frac{L}{a}\right)$$

$$= (J\pi - 2T) \operatorname{Log}\left[\frac{L}{a}\right]$$

• $J\pi > 2T$ (Low temperature)

- $\underline{J\pi > 2T}$ (low temperature)

  $F_v > 0 \quad \Rightarrow$ $\underline{\underline{\text{No}}}$ vortices

  only spin wave configurations

  $$\langle S_r^x S_0^x \rangle = \left(\frac{a}{r}\right)^{cT}$$

- $\underline{J\pi < 2T}$ $\quad T_c \text{ "=" } \left(\frac{J\pi}{2}\right)$

  Favorable to create vortices $\Rightarrow$ n_{vortex} will be finite

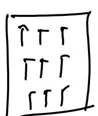
$$\langle S_r^x S_0^x \rangle \sim e^{-r/\xi}$$

Phase transition.

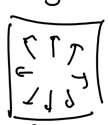
- Between a phase $\begin{cases} (1/r)^T \\ e^{-r/\xi}. \end{cases}$

- $\underline{\text{No}}$ ordered state $\quad T < T_c \quad T > T_c$

- Phase are distinguished by their topological nature

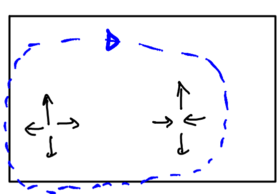


no free vortices



free. vortices.

$\underline{\text{treat the system with many vortices.}}$



$$\oint_C d\theta = 0$$

$$\theta(r) = \psi(r) + \tilde{\theta}(r)$$

| ψ : smooth part.
| $\tilde{\theta}$: configuration that potentially can contain. vortices.

$$\oint d\theta = 2\pi n.$$

$$\oint_C d\tilde{\theta} = 2\pi n.$$

If not on the vortex $\quad E = \frac{J}{2} \int d^2r \, (\nabla\tilde{\theta})^2.$

Minimize $E \qquad \Delta\tilde{\theta} = 0$

$$E[\tilde\theta] = \frac{J}{2}\int d^2r \quad \nabla_x\tilde\theta\,\nabla_x\tilde\theta + \nabla_y\tilde\theta\,\nabla_y\tilde\theta$$

$$E[\tilde\theta + \delta\tilde\theta] = \frac{J}{2}\int d^2r\,\left[\nabla_x\tilde\theta\,\nabla_x\tilde\theta + \nabla_y\tilde\theta\,\nabla_y\tilde\theta\right]$$
$$+ J\int d^2r\,\left[\nabla_x\tilde\theta\,\nabla_x\delta\tilde\theta + \nabla_y\tilde\theta\,\nabla_y\delta\tilde\theta\right]$$

$$E[\tilde\theta + \delta\tilde\theta] - E[\tilde\theta] = J\int d^2r\,\left[\nabla_x\tilde\theta\,\nabla_x\delta\tilde\theta + \nabla_y\tilde\theta\,\nabla_y\delta\tilde\theta\right]$$
$$= -J\int d^2r\;\delta\tilde\theta\left[\nabla_x^2\tilde\theta + \nabla_y^2\tilde\theta\right]$$

$$\nabla_x^2\tilde\theta + \nabla_y^2\tilde\theta = 0 \qquad \Delta\tilde\theta = 0$$

$$f(z) = a(z) + i\,b(z) \qquad a(z) = \mathrm{Re}\,f \qquad b(z) = \mathrm{Im}\,f$$

if f is analytic.
$$\begin{cases}\partial_x a(z) = \partial_y b(z)\\ \partial_y a(z) = -\partial_x b(z)\end{cases}$$

$$f(z) = (x+iy)^2 \equiv z^2 = x^2 - y^2 + 2ixy.$$

$$f(z) = \tilde\theta(x,y) + i\,\bar\theta(x,y)$$

$$\int d\tilde\theta = \int d\vec{l}\cdot\vec\nabla\tilde\theta \qquad \begin{vmatrix}\partial_x\tilde\theta\\ \partial_y\tilde\theta\end{vmatrix} = \begin{vmatrix}\partial_y\bar\theta\\ -\partial_x\bar\theta\end{vmatrix}$$

$$\int d\tilde\theta = \int d\vec{l}\cdot\mathrm{rot}\,\bar\theta = \int d\vec{S}\;\mathrm{rot}\,\mathrm{rot}\,\bar\theta$$
$$= -\int d\vec{S}\;\Delta\bar\theta = 2\pi n.$$

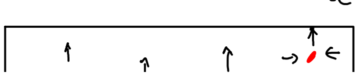
$$\Delta\bar\theta(x,y) = 2\pi n.\,\delta(r - r_i)$$

r_i is the position of the vortex core

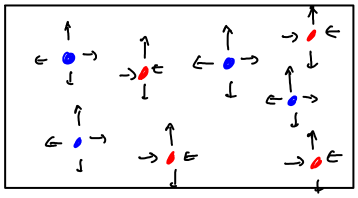
$$\Delta V(r) + \frac{\rho(r)}{\varepsilon_0} = 0 \qquad \text{Poisson Equation.}$$

$\bar\theta$ is the (two dimensional) Coulomb potential of a charge n. put at the position r_i

$$\bar\theta(r) = 2\pi n.\,\mathrm{Log}\left[\frac{r - r_i}{a}\right]$$



$$\bar\theta(r) = \sum 2\pi n_i\,\mathrm{Log}\left[r - r_i\right]$$



$$\bar{\theta}(r) = \sum_i 2\pi n_i \,\mathrm{Log}\left[\frac{r-r_i}{a}\right]$$

$$E = \frac{J}{2}\int d^2r \left[\nabla_x \tilde{\theta}\, \nabla_x \tilde{\theta} + \nabla_y \tilde{\theta}\, \nabla_y \tilde{\theta}\right]$$

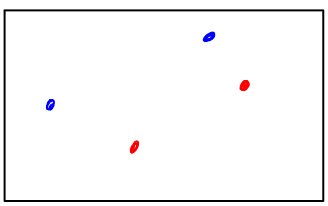
$$= \frac{J}{2}\int d^2r \left[\nabla_y \bar{\theta}\, \nabla_y \bar{\theta} + \nabla_x \bar{\theta}\, \nabla_x \bar{\theta}\right]$$

$$= -\frac{J}{2}\int d^2r \left[\bar{\theta}\, \nabla_y^2 \bar{\theta} + \bar{\theta}\, \nabla_x^2 \bar{\theta}\right]$$

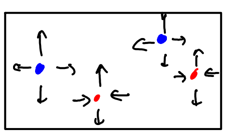
$$= -\frac{J}{2}\int d^2r \left[\bar{\theta}(r)\; \Delta\bar{\theta}(r)\right]$$

$$\bar{\theta}(r) = \sum_i 2\pi n_i \,\mathrm{Log}\left[\frac{r-r_i}{a}\right] \qquad \Delta\bar{\theta}(r) = -\sum_j 2\pi n_j\, \delta(r-r_j)$$

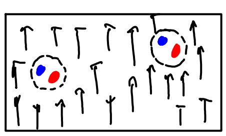
$$= \frac{J}{2}\sum_{ij} (2\pi n_i)(2\pi n_j)\, \mathrm{Log}\left[\frac{r_i-r_j}{a}\right]$$



$$Z_4 = \int dr_1\, dr_2\, dr_3\, dr_4\; e^{\beta \sum_{ij} n_i n_j \mathrm{Ln}\left[\frac{r_i-r_j}{a}\right]}$$



free vortices. (high temperature)

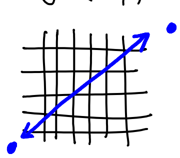


bound vortex pairs. (low temperatures)

III] Renormalization:

- $H_{\text{microscopic}} = -J\sum_{\langle ij\rangle} \vec{S}_i \cdot \vec{S}_j$

$$H = \int (\nabla\psi)^2 + a|\psi|^2 + b|\psi|^4.$$



a.

$\boxed{r \gg a}$ $\qquad \langle S_r\, S_0\rangle$

- $\Lambda \approx \pi/a$ $\qquad$ (Energies. $\ll \Lambda$

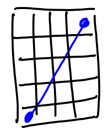
$L\, r, t \gg a\,, \; v a$

$$H_{SG} = \frac{1}{2\pi}\int dx \left[uK(\pi\Pi)^2 + \frac{u}{K}(\nabla\varphi)^2 \right] - V_0 \int dr \cos(\sqrt{2}\varphi)$$

perturbation in V_0.

$$Z = Z_0 \Big[1 - \underbrace{V_0^2 \int^L d^2r_1\, d^2r_2 \left(\frac{1}{r_1 - r_2} \right)^{\nu}}_{\text{potentially divergent!}} + \cdots \Big]$$

$$f(r) \equiv 1 - V_0^2 \operatorname{Log}\left[\frac{r}{a}\right] + V_0^4$$



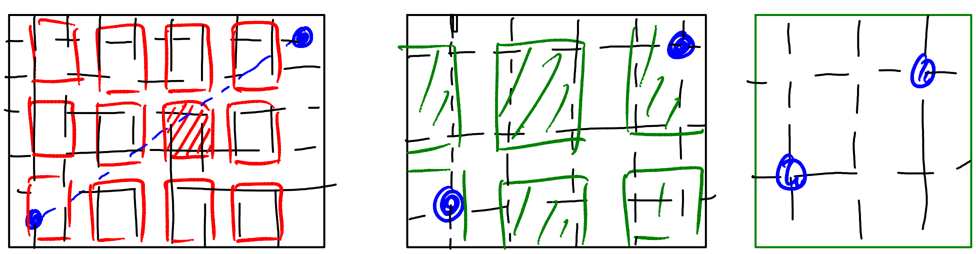
r is fixed $\rightarrow$ perturbation is OK.

fix V_0 look at the system when $r \to \infty$

* Wilson. (1971 PRB 4 3174)

$H[V_0, V_1 ..]$	$H[V_0', V_1', V_2', \ldots]$
microscopic cutoff a.	larger microscopic cutoff $a' > a$.
Look at r	Look at r

have the same physics at large distances.



$H[V_1]$	$\rightarrow$	$H[V_2]$	$\rightarrow$	$H[V_3]$	$\rightarrow$	$H[V_4]$
a_1	$<$	a_2	$<$	a_3	$<$	a_4

\# Renormalization group.

- just relate problems with the same physics.
- H_i exactly solvable.
- $a_i \nearrow$ growing. $r \approx a_i \Rightarrow$ perturbation theory is now well controlled.

$H[\;]$ reflects more and more the large is now well controlled.

— $H_i[\ ]$ reflect more and more the large lengthscale properties : (approximations)

2) Example of sine Gordon model.

$$H = \frac{1}{2\pi}\int dx\ uK\,(\pi\Pi)^2 + \left(\frac{u}{K}\right)(\nabla\varphi)^2 - V_0\int dx\ \cos(\sqrt{2}\,\phi)$$

$$S = \frac{1}{2\pi K}\int dx\,d\tau\left[\frac{1}{u}(\partial_\tau\varphi)^2 + u\,(\partial_x\varphi)^2\right] - V\int dx\,d\tau\ \cos(\sqrt{2}\varphi)$$

$y = u\tau$ $\quad$ y : dimension of a distance. $\quad \beta \to \infty$.

$$\frac{1}{2\pi K}\int dx\,d\tau\ u\,(\partial_y\varphi)^2 + u\,(\partial_x\varphi)^2$$

$$S = \frac{1}{2\pi K}\int dx\,dy\ \left[(\partial_y\varphi)^2 + (\partial_x\varphi)^2\right] - \underbrace{\left(\frac{V}{u}\right)}_{g}\int dx\,dy\ \cos(\sqrt{2}\varphi)$$

$$S_0 = \frac{1}{2\pi K}\int dx\,dy\ \left[(\partial_y\varphi)^2 + (\partial_x\varphi)^2\right]$$

$$\langle e^{i\sqrt{2}\,\varphi(x,y)}\, e^{-i\sqrt{2}\,\varphi(0,0)}\rangle = e^{\frac{1}{2}\langle[\quad]^2\rangle} = e^{-\langle[\varphi(x,y)-\varphi(0,0)]^2\rangle}$$

$$\varphi = \frac{1}{L_x L_y}\sum_{q\omega} e^{i(qx-\omega y)}\ \varphi_{q\omega}.$$

$$S_0 = \frac{1}{2\pi K}\int d^2r\ \frac{1}{\Omega^2}\sum_{\substack{q_1\omega_1\\ q_2\omega_2}} e^{i(\ )_1}\, e^{-i(\ )_2}\ \varphi^*_{q_1\omega_1}\varphi_{q_2\omega_2}\left[q_1q_2 + \omega_1\omega_2\right]$$

$$= \frac{1}{2\pi K}\,\frac{1}{\Omega}\sum_{q\omega}\varphi^*_{q\omega}\varphi_{q\omega}\left[q^2+\omega^2\right]$$

$$\langle[\varphi(r)-\varphi(0)]^2\rangle = \frac{1}{\Omega^2}\sum_{\vec q_1\vec q_2}\left(e^{i\vec q_1\vec r}-1\right)\left(e^{i\vec q_2\vec r}-1\right)\langle\varphi_{\vec q_1}\varphi_{\vec q_2}\rangle$$

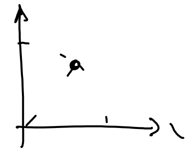
$$= \frac{1}{\Omega}\sum_{\vec q}\left(e^{i\vec q\vec r}-1\right)\left(e^{-i\vec q\vec r}-1\right)\ \frac{\pi K}{\|\vec q\|^2}$$

$$= \frac{2\pi K}{\Omega}\sum_{\vec q}\left(1-\cos(\vec q\cdot\vec r)\right)\ \frac{1}{\|q\|^2}.$$

$$\Lambda \sim 1/a.$$

$$2\pi K\int d^2q\ (1-\cos(\vec q\cdot\vec r))\ \frac{1}{\ }$$

$$= \frac{2\pi K}{(2\pi)^2} \int_0 d^2q \, (1-\cos(\vec{q}\cdot\vec{r})) \frac{1}{q^2}$$



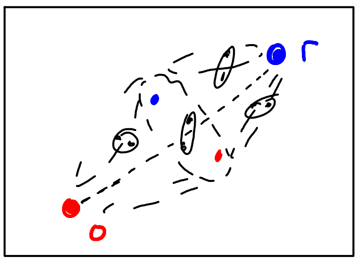
$$= \frac{K}{2\pi} \int_0^\Lambda \frac{d^2q}{q^2} \left(1-\cos(\vec{q}\cdot\vec{r})\right)$$

$$\int_0^{+\infty} \frac{d^2q}{q} (1-\cos \vec{q}\cdot\vec{r}) \, e^{-q/\Lambda}$$

$$\approx \frac{K}{2\pi} \int_{1/r}^\Lambda \frac{d^2q}{q^2} = K \int_{1/r}^\Lambda \frac{dq}{q}$$

$$= K \operatorname{Log}\left[\frac{r}{a}\right] \qquad r = \sqrt{x^2+y^2}$$

$$S_0 \qquad \left\langle e^{i\sqrt{2}\varphi(r)} e^{-i\sqrt{2}\varphi(0)} \right\rangle = e^{-K \operatorname{Log}\left[\frac{r}{a}\right]}$$



$$Q = \sqrt{K} \qquad Q = -\sqrt{K}.$$

Screening by dipoles will change the potential.

$$\left\langle e^{i\sqrt{2}\varphi(r)} e^{-i\sqrt{2}\varphi(0)} \right\rangle_{S=S_0+S_v} \qquad e^{-K\operatorname{Log}[r/a]}$$

Potential between two test charges.

$$S = S_0 + S_v$$

$$\left\langle e^{i\sqrt{2}\varphi(r)} e^{-i\sqrt{2}\varphi(0)} \right\rangle = \frac{\int \mathcal{D}\varphi \, e^{-S_0-S_v} \, e \, e}{\int \mathcal{D}\varphi \, e^{-S_0-S_v}}$$

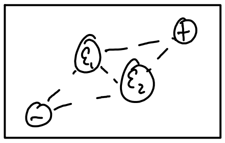
$$= \frac{\int \mathcal{D}\varphi \, e^{-S_0} \, e \, e \left[1 - S_v + \frac{1}{2} S_v^2\right]}{\int \mathcal{D}\varphi \, e^{-S_0} \left[1 - S_v + \frac{1}{2} S_v^2\right]}$$

$$= \frac{\left\langle e^{i(\,)} e^{-i(\,)} \right\rangle_{S_0} + \frac{1}{2} g^2 \int d^2r_1 \, d^2r_2 \left\langle e^{i\sqrt{2}\varphi(r)} e^{-i\sqrt{2}\varphi(0)} \cos(\sqrt{2}\varphi(r_1)) \cos(\sqrt{2}\varphi(r_2)) \right\rangle}{1 + \frac{1}{2} g^2 \int d^2r_1 \, d^2r_2 \left\langle \cos(\sqrt{2}\varphi(r_1)) \cos(\sqrt{2}\varphi(r_2)) \right\rangle_0}$$

$$= \langle \quad \rangle_{S_0} + \frac{1}{2} g^2 \int d^2r_1 \, d^2r_2 \left[\left\langle e^{i\sqrt{2}} e^{-i\sqrt{2}} \cos(\,)\cos(\,) \right\rangle_0 - \left\langle e^{i\sqrt{2}} e^{-i\sqrt{2}} \right\rangle_0 \left\langle \cos(\,)\cos(\,) \right\rangle_0 \right]$$

$$-K \operatorname{Log}\left[\frac{r}{a}\right] \quad \ldots \quad e^{i\sqrt{2}\varphi(r)} \, e^{-i\sqrt{2}\varphi(0)} \quad e^{i\sqrt{2}\,\varepsilon\,\varphi(r_1)} \, e^{-i\sqrt{2}\,\varepsilon\,\varphi(r_2)}$$

$$= e^{-K\operatorname{Log}\left[\frac{r}{a}\right]} + \frac{1}{8} g^2 \sum_{\varepsilon_1 \varepsilon_2} \int d^2 r_1 d^2 r_2 \left[\left\langle e^{i\sqrt{2}\varphi(r)} e^{-i\sqrt{2}\varphi(0)} e^{i\sqrt{2}\varepsilon_1 \varphi(r_1)} e^{-i\sqrt{2}\varepsilon_2 \varphi(r_2)} \right\rangle_0 - \langle \quad \rangle_0 \langle \quad \rangle_0 \right]$$



$$= e^{-K\operatorname{Log}\left[\frac{r}{a}\right]} + \frac{1}{8} g^2 \sum_{\varepsilon} \int d^2 r_1 d^2 r_2 \left[\left\langle e^{i\sqrt{2}\varphi(r)} e^{-i\sqrt{2}\varphi(0)} e^{i\sqrt{2}\varepsilon\varphi(r_1)} e^{-i\sqrt{2}\varepsilon\varphi(r_2)} \right\rangle_0 - \underbrace{\langle \quad \rangle}_{e^{-K\operatorname{Log}[r/a]}} \underbrace{\langle \quad \rangle}_{e^{-K\operatorname{Log}\left[\frac{r_1-r_2}{a}\right]}} \right]$$

$$\left\langle e^{i\sqrt{2}\varphi(r) - i\sqrt{2}\varphi(0) + i\varepsilon\sqrt{2}\varphi(r_1)} e^{-i\sqrt{2}\varphi(r_2)} \right\rangle_0 = e^{-\left\langle \left[\varphi(r) - \varphi(0) + \varepsilon\varphi(r_1) - \varepsilon\varphi(r_2) \right]^2 \right\rangle_0}$$

$$e^{-\left[\left(\underbrace{e^{iqr} - 1} + \varepsilon e^{iqr_1} - \varepsilon e^{iqr_2} \right) \left(\underbrace{e^{-iqr} - 1} + \varepsilon e^{-iqr_1} - \varepsilon e^{-iqr_2} \right) \langle \varphi_q \varphi_q \rangle_0 \right.}$$

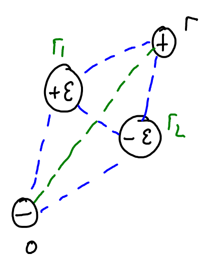
$$\left[\underbrace{\left(e^{iqr} - 1\right)\left(e^{-iqr} - 1\right)}_{[2 - 2\cos(qr)]} + \varepsilon^2 \underbrace{\left(e^{iqr_1} - e^{iqr_2}\right)\left(e^{-iqr_1} - e^{-iqr_2}\right)}_{[2 - 2\cos(q(r_1 - r_2))]} \right.$$
$$\left. + \varepsilon \left(e^{iqr} - 1\right)\left(e^{-iqr_1} - e^{-iqr_2}\right) + \left(e^{iqr_1} - e^{iqr_2}\right)\left(e^{-iqr} - 1\right) \right]$$

(iii) $\varepsilon \left[2\cos(q(r - r_1)) - 2\cos(q(r - r_2)) - 2\cos(q(0 - r_1)) + 2\cos(q(0 - r_2)) \right]$

$$\longrightarrow \quad K \operatorname{Log}[r/a] + K \operatorname{Log}\left[\frac{r_1 - r_2}{a}\right]$$

(iii) $\varepsilon \left[\left(2\cos(q(r - r_1)) - 2\right) - \left(2\cos(q(r - r_2)) - 2\right) - \left(2\cos(q(0 - r_1)) - 2\right) + \left(2\cos(0 - r_2) - 2\right) \right]$

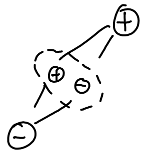
$$K \operatorname{Log}\left[\frac{r - 0}{a}\right] + \varepsilon^2 K \operatorname{Log}\left[\frac{r_1 - r_2}{a}\right] + \varepsilon K \left[\operatorname{Ln}\left[\frac{r - r_2}{a}\right] - \operatorname{Ln}\left[\frac{r - r_1}{a}\right] + \operatorname{Ln}\left[\frac{0 - r_1}{a}\right] - \operatorname{Ln}\left[\frac{0 - r_2}{a}\right] \right]$$



$$\sum_{i,j} n_i n_j \operatorname{Log}\left[\frac{r_i - r_j}{a}\right]$$

$$e^{-K\operatorname{Log}[r/a]} + \frac{1}{8} g^2 \sum_{\varepsilon} \int d^2 r_1 d^2 r_2 \; e^{-K\operatorname{Log}\left[\frac{r}{a}\right]} e^{-K\operatorname{Log}\left[\frac{r_1 - r_2}{a}\right]} \left(e^{\varepsilon K \left[\operatorname{Log}\left[\frac{r - r_1}{a}\right] - \operatorname{Ln}\left[\frac{r - r_2}{a}\right] + \operatorname{Ln}\left[\frac{0 - r_2}{a}\right] - \operatorname{Ln}\left[\frac{0 - r_1}{a}\right] \right]} - 1 \right)$$

$$\langle \quad \rangle_S = e^{-K\,\mathrm{Log}[r/a]}\left(1+\frac{1}{8}g^2\sum_\varepsilon \int_{|r_1-r_2|>a} d^2r_1\, d^2r_2 \quad e^{-K\,\mathrm{Ln}\left[\frac{r_1-r_2}{a}\right]}\right.$$



$$\left(e^{\varepsilon K\,\mathrm{Ln}\left[\frac{r-r_1}{a}\right]-\mathrm{L}\left[\frac{r-r_2}{a}\right]+\mathrm{Ln}\left[\frac{0-r_2}{a}\right]-\mathrm{Log}\left[\frac{0-r_1}{a}\right]}-1\right)$$

$$\vec{R}=\frac{\vec{r_1}+\vec{r_2}}{2} \qquad u=\vec{r_1}-\vec{r_2} \qquad \begin{cases} r_1=\vec{R}+\vec{u}/2 \\ r_2=\vec{R}-\vec{u}/2 \end{cases}$$

$$F(x)=\mathrm{Ln}\left[\frac{x}{a}\right]$$

$$F[r-r_1]-F[r-r_2]+F[0-r_2]-F[0-r_1]$$

$$F\left[r-R-\frac{u}{2}\right]-F\left[r-R+\frac{u}{2}\right]+F\left[0-R+\frac{u}{2}\right]-F\left[0-R-\frac{u}{2}\right]$$

$$\left[\frac{u}{2}\nabla F[r-R]-\left(\frac{-u}{2}\right)\nabla F[r-R]+\frac{u}{2}\nabla F[0-R]-\left(\frac{-u}{2}\right)\nabla F[0-R]\right]$$

$$=\vec{u}.\vec{\nabla}\cdot\Big[F[r-R]-F[0-R]\Big]$$

$$\sum_\varepsilon\int d^2R\, d^2u \quad e^{-K F(u)}\left(e^{\varepsilon K\, \underbrace{\vec{u}.\vec{\nabla}}_{[u_x\partial_x+u_y\partial_y]}\,\big[F[r-R]-F[0-R]\big]}-1\right)$$

$$\left(e^{\varepsilon\cdots}-1\right) = \frac{\varepsilon^2K^2}{2}\Big(\left(u_x\nabla_x+u_y\nabla_y\right)\big[F(r-R)-F(0-R)\big]\Big)^2$$

$$\int du_x du_y \quad e^{-K\,\mathrm{Ln}\left[\frac{\sqrt{u_x^2+u_y^2}}{a}\right]}$$ $\rightarrow$ invariance by rotation in u

$$u_x\nabla_x[\quad]\,u_x\nabla_x[\quad] + u_y\nabla_y[\quad]\,u_y\nabla_y[\quad]$$
$$+\cancel{2u_x\nabla_x[\quad]\,u_y\nabla_y[\quad]}$$ (integral over u)

$$\int d^2u\; e^{-KF(u)}\, u_x^2 = \int\cdots u_y^2 = \frac{1}{2}\int d^2u\; e^{-KF(u)}\,u^2$$

$$\frac{\varepsilon^2}{2}\frac{1}{2}\int d^2R\int d^2u\quad e^{-K\,\mathrm{Log}\left[\frac{u}{a}\right]}\; u^2 \quad \nabla_x\big[F[r-R]-F[0-R]\big]\nabla_x[\quad] + \nabla_y[\quad]\nabla_y[\quad]$$

$$\frac{1}{16}g^2K^2\left(\int d^2u\; u^2\, e^{-K\,\mathrm{Ln}\left[\frac{u}{a}\right]}\right)$$
$$\left(\int d^2R\;\Big(\nabla_x\big[F(r-R)-F(0-R)\big]\nabla_x[\quad]+\nabla_y[\quad]\nabla_y[\quad]\Big)\right)$$

$$-\big[F(r-R)-F(0-R)\big]\nabla^2[\quad] - [\quad]\nabla_y^2[\quad]$$

$$- \left[F(r-R) - F(0-R)\right] \nabla_x^2 [\quad] - [\quad] \nabla_y^2 [\quad]$$

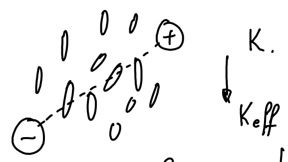
$$\int - \left[F(r-R) - F(0-R)\right] \Delta \left[F[r-R] - F[0-R]\right] d^2R.$$

$$- \int \left[F[r-R] - F[0-R]\right] \left[(2\pi)\,\delta(r-R) - (2\pi)\,\delta(0-R)\right] d^2R.$$

$$4\pi\, F[0-r] = (4\pi) \log[r/a]$$

$$\langle \quad \rangle_S = e^{-K\log[r/a]} \left(1 + \frac{(4\pi)}{16} g^2 K^2 \log[r/a] \int_a^{(r)} d^2u\, u^2 e^{-K\log\left[\frac{u}{a}\right]}\right),$$

$$= e^{-K_{eff}\log[r/a]}$$



Screening by the dipoles.

$$K_{eff} = K - \frac{\pi}{4} g^2 K^2 \int_a^{+\infty;\, r} d^2u\; u^2\; e^{-K\log\left[\frac{u}{a}\right]}$$

$$K_{eff} = K - \frac{2\pi^2}{4} g^2 K^2 \int_a^{+\infty} du\; u^3 \left(\frac{a}{u}\right)^K.$$

$$K_{eff} = K - \frac{\pi^2}{2} g^2 K^2 a^4 \int_a^{+\infty} \frac{du}{a} \left(\frac{u}{a}\right)^3 \left(\frac{a}{u}\right)^K.$$

$$K_{eff} = K - \frac{\pi^2}{2} \underbrace{(g^2 K^2 a^4)}_{y^2 K^2} \int_a^{+\infty} \frac{du}{a} \left(\frac{u}{a}\right)^{3-K}.$$

$$K_{eff} = K - \frac{\pi^2}{2} y^2 K^2 \int_a^{+\infty,\, r} \frac{du}{a} \left(\frac{u}{a}\right)^{3-K}.$$

$a \to a' > a.$ $\qquad a' = a + da.$

Physics invariant $\Rightarrow$ K_{eff} remains <u>unchanged</u>

$$K_{eff} = K(a) - \frac{\pi^2}{2} y^2(a) K^2 \int_a^{a'=a+da} \frac{du}{a} \left(\frac{u}{a}\right)^{3-K}.$$

$$- \frac{\pi^2}{2} y^2(a) K^2 \int_{a'}^{+\infty} \frac{du}{a} \left(\frac{u}{a}\right)^{3-K}.$$

$$K_{eff} = K(a') - \frac{\pi}{2} y^2(a') K^2 \int_{a'}^{+\infty} \frac{du}{a'} \left(\frac{u}{a'}\right)^{3-K}.$$

$$\begin{cases} K(a') = K(a) - \dfrac{\pi^2}{2} y^2(a) K^2 \dfrac{da}{a}. \\ y^2(a') = y^2(a) \left(\dfrac{a'}{a}\right)^{4-K}. \end{cases}$$

$a \to a + da. = a'$

$a = a_0 e^{\ell}.$

$\ell = 0 \qquad a(\ell) = a_0.$

$da = a_0 e^{\ell} d\ell.$

$\frac{da}{a} = d\ell.$

$$y^2(a+da) = y^2(a) \left(1 + \frac{da}{a}\right)^{4-K}.$$

$$K(a+da) - K(a) = -K^2 \frac{\pi^2}{2} y^2(a) \quad d\ell.$$

$da \dfrac{dK}{da}.$

$$\frac{\partial K}{\partial \ell} = -\frac{\pi^2}{2} y^2(a)$$

$$y^2(\ell + d\ell) = y^2(\ell)\, e^{(4-K)d\ell.}$$
$$= y^2(\ell) - + (4-K)\, y^2(\ell)\, d\ell$$

$$\frac{\partial y^2(\ell)}{\partial \ell} = (4-K)\, y^2(\ell). \Rightarrow \frac{\partial y(\ell)}{\partial \ell} = \frac{1}{2}(4-K)\, y(\ell)$$

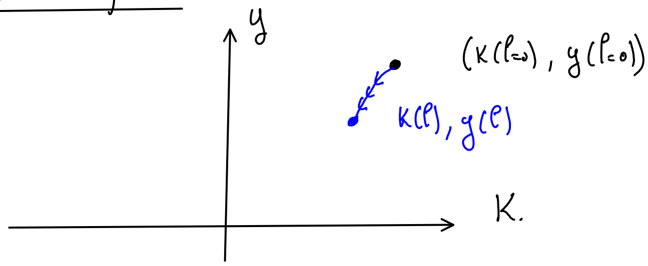
$$\boxed{\begin{cases} \dfrac{\partial K}{\partial \ell}(\ell) = -y^2(\ell) K^2 \\ \dfrac{\partial y(\ell)}{\partial \ell} = \dfrac{1}{2}(4-K)\, y(\ell) \end{cases}}$$

BKT equations.

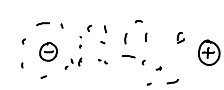
$\begin{cases} K: \text{ strength of the } \text{Log}[r/a] \text{ potential.} \\ y: \text{ fugacity of the vortices} \end{cases}$

Start with $K(\ell=0) \quad y(\ell=0) \quad a(\ell=0) = a_0$

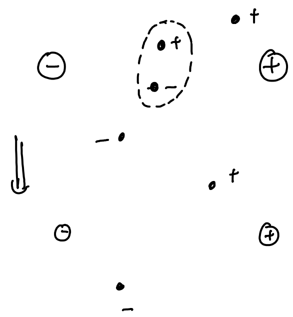
Study of the flow.



$\dfrac{\partial K}{\partial \ell} = -y^2 . Cst. \qquad K \searrow$ when $\ell \nearrow$



Screen the charges.: decreasing the effective potential between them.



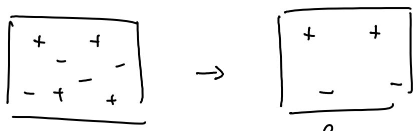
eliminating dipoles between a and a'

$K(\ell=0)$

$\downarrow$

$K(\ell)$

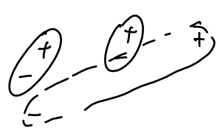
- $\dfrac{\partial y(\ell)}{\partial \ell} = \dfrac{1}{2}(4-K)\, y(\ell).$



$\ell = 0$ $\quad$ $y(\ell=0)$ $\qquad\qquad$ $\ell \neq 0.$ $\quad$ $y(\ell)$

$K > 4 \qquad y(\ell) \searrow$

K very large $\qquad V \equiv K \log[r/a]$



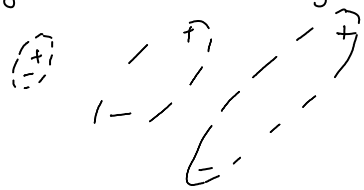
+ and − charges attract strongly

$\Rightarrow$ Mostly small pairs are created

$\Rightarrow$ fugacity for large dipoles is weaker.

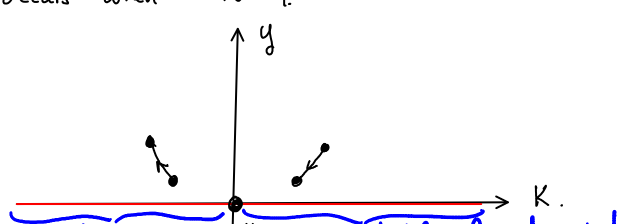
K very small :
+ and − charges are very weakly bound.

$\Rightarrow$ easy to create arbitrarily large pairs of charges.



$y(\ell) \nearrow$

Something occurs when. $K = 4.$



$(y(\ell=0), K)$: fixed points (fixed line) of the equations

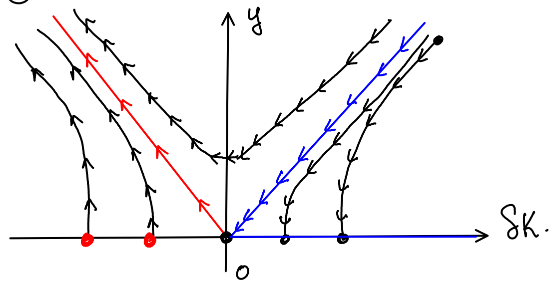
$K = 4 + 2\delta K.$

$$\begin{cases} 2\dfrac{\partial \delta K}{\partial \ell} = -y^2\, 16 \\ \partial y \cdot = -\tfrac{1}{2} 2 \delta K\, y \end{cases} \qquad \begin{aligned} \frac{\partial \delta K}{\partial \ell} &= -8y^2 \\ \frac{\partial y}{} &= -\delta K . y \end{aligned}$$

$$\begin{cases} \frac{\partial \cdots}{\partial e} = -y \cdots \\ \frac{\partial y}{\partial e} = -\frac{1}{2} 2\delta K\, y . \end{cases} \qquad \frac{\partial \cdots}{\partial e} = -8y \qquad \frac{\partial y}{\partial e} = -\delta K . y .$$

$$\tilde{y} = \sqrt{8}\, y$$

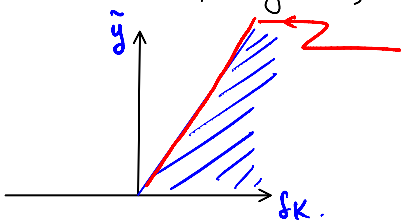
$$\begin{cases} \frac{\partial \delta K}{\partial e} = -\tilde{y}^2 \\ \frac{\partial \tilde{y}}{\partial e} = -\delta K \tilde{y} \end{cases}$$



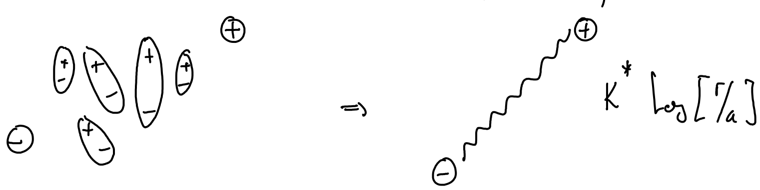
$$\frac{\partial \delta K}{\partial \tilde{y}} = \frac{-\tilde{y}^2}{-\delta K \tilde{y}} \iff \delta K\, \partial \delta K = \tilde{y}\, \partial \tilde{y}$$

$$\delta K^2 = \tilde{y}^2 + \text{cste}$$

$$A^2 = \delta K^2(\ell=0) - \tilde{y}^2(\ell=0) \qquad \delta K^2 - \tilde{y}^2 = A^2 \quad \text{(invariant of the flow)}$$



Physics:
$$\begin{cases} \text{no free vortices.} \\ \text{Renormalized strength of the potential } K^* \end{cases}$$



Correlation functions.

$$\langle S^x_r S^x_0 \rangle \leftrightarrow \langle e^{i\sqrt{2}\varphi_r} e^{-i\sqrt{2}\varphi_0} \rangle \sim e^{-K^* \mathrm{Log}[r/a]}$$

$$\sim \left(\frac{a}{r}\right)^{K^*}$$

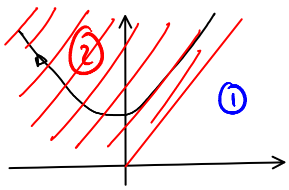
⇒ power law correlation functions.

[XY: only spin waves no free vortices.
Coulomb: finite dielectric constant (unscreened charges)

Coulomb: finite dielectric constant (unscreened charges)

Sine-Gordon: Quadratic only $\to$ power law correlation

[renormalized K^*]

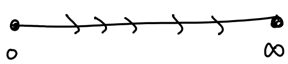


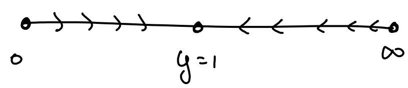
$$\frac{\partial K}{\partial \ell} = -y^2 + y^4 + y^6$$
$$\frac{\partial y^2}{\partial \ell} = (4-K)\, y^2 + y^4 + y^6$$

1 The perturbative RG equations are becoming more and more accurate

2 $y \to \infty$ from second order RG.
Perturbative RG equations are becoming increasingly wrong.
Should not be trusted beyond $y(\ell) \simeq 1$

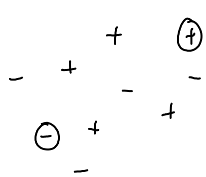
$$\frac{\partial y}{\partial \ell} = y^2 . \qquad y \to \infty \qquad\qquad \frac{\partial y}{\partial \ell} = y^2 - y^3$$





$$S = \frac{1}{2\pi K} \int dx\, d\tau \; \frac{1}{u}(\partial_\tau \phi)^2 + u(\nabla\phi)^2 \; - \; V_0 \int dx\, d\tau \cos(\sqrt{2}\,\phi)$$

2 K decreasing. ($\to 0$??) $y \to V_0$ increasing.
[More and more easy to create vortices]



arbitrarily large pairs $\Rightarrow$ free vortices (free charges)

$$\phi = 0 + \bar\phi$$
$$S \approx \frac{1}{2\pi K(\ell^*)} \int dx\, d\tau \; \frac{1}{u}(\partial_\tau \bar\phi)^2 + u(\partial_x \bar\phi)^2 + \frac{V_0(\ell^*)}{2} \int dx\, d\tau \; \bar\phi^2(x,\tau)$$

$$a \longrightarrow a(\ell^*) \text{ for which } y(\ell^*) \simeq O(1)$$

$$S = \frac{1}{2\pi K^* \Omega} \sum_{q,\omega} \left(\frac{1}{u}\omega_n^2 + u q^2\right) \phi^* \phi + \frac{V_0}{2} \frac{1}{\Omega} \sum_{q\omega} \phi^* \phi .$$

$$\langle \phi^*_{q\omega_n} \phi_{q\omega_n} \rangle \simeq \frac{K}{\omega_n^2 + q^2 + V_0}$$

$$\langle [\phi(r,\tau) - \phi(0,0)]^2 \rangle \equiv \frac{1}{\Omega} \sum_{q\omega_n} \frac{[2 - 2\cos(qr + \omega\tau)]}{\omega_n^2 + q^2 + V_0}$$

Convergent at small q

$$\langle S^y_r S^y_0 \rangle \equiv \langle \sin(\phi(\vec{r})) \sin(\phi(0)) \rangle \qquad \vec{r} = (x,\tau)$$

$$\frac{1}{(2i)^2} \left[e^{i\phi} e^{i\phi} + e^{-i\phi} e^{-i\phi} - e^{i\phi} e^{-i\phi} - e^{-i\phi} e^{i\phi} \right]$$

$$= \frac{2}{(2i)^2} \left[e^{-\frac{1}{2} \langle [\phi(x,\tau) + \phi(0,0)]^2 \rangle} - e^{-\frac{1}{2} \langle [\phi(x,\tau) - \phi(0,0)]^2 \rangle} \right]$$

$$= \frac{+1}{2} \left[e^{-\frac{1}{2} \frac{1}{\Omega} \sum_{q\omega} \frac{2(1 - \cos(\omega\tau + qx))}{\omega^2 + q^2 + V_0}} - e^{-\frac{1}{2\Omega} \sum_{q\omega_n} \frac{2(1 + \cos(qx + \omega\tau))}{\omega^2 + q^2 + V_0}} \right]$$

$$= \frac{1}{2} \left[e^{-\frac{K}{\Omega} \sum_k \frac{1 - \cos(\vec{k}\vec{r})}{k^2 + V_0}} - e^{-\frac{K}{\Omega} \sum_k \frac{1 + \cos(\vec{k}\vec{r})}{k^2 + V_0}} \right]$$

$V_0 = 0 \qquad \langle \sin(\phi(r)) \sin(\phi(0)) \rangle \simeq \left(\frac{a}{r}\right)^K$

$V \neq 0 \qquad r \to \infty \qquad e^{-\frac{K}{\Omega} \sum_k \frac{1}{k^2 + V_0}} - e^{-\frac{K}{\Omega} \sum_k \frac{1}{k^2 + V_0}} = 0$

$$\langle \quad \rangle = \frac{1}{2} \left[\frac{2K}{\Omega} \sum_k \frac{\cos(\vec{k}\vec{r})}{k^2 + V_0} \right]$$

$$= \frac{K}{\Omega} \sum_k \frac{\cos(\vec{k} \cdot \vec{r})}{k^2 + V_0} \qquad \text{Screened Coulomb potential}$$

$$\sim e^{-r/\xi} \qquad k^2 \equiv V_0 \equiv 1/\xi^2$$

V_0 gives the Screening length of the potential.

(phase I) $\to$
- power law correlations. $\langle \sin(\phi)\sin(\phi) \rangle \sim \left(\frac{1}{r}\right)^{K^*}$
- finite dielectric.
- Quadratic action. (Massless theory)

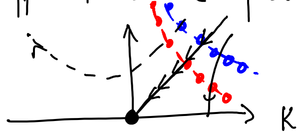
(phase II) $\to$
- exponential correlations $\langle \quad \rangle \sim e^{-r/\xi}$
- Screening (free charges)

| Massive theory.

Characteristic of BKT

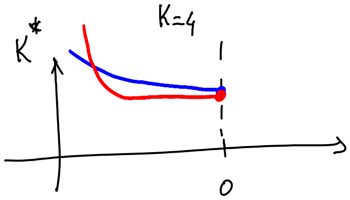
$$S_0 = \underbrace{\frac{1}{2\pi K^*}} \int dx\,d\tau \, (\partial_\tau \phi)^2 + (\partial_x \phi)^2 .$$

Coefficient in the quadratic action.



Right at the transition.

$K \to K^* = 4$ universal value.



δ : distance to the transition.

At the transition <u>universal</u> value for K^*

Correlation Length :

$$\langle \sin(\phi_r) \sin(\phi_0) \rangle \sim e^{-r/\xi}$$

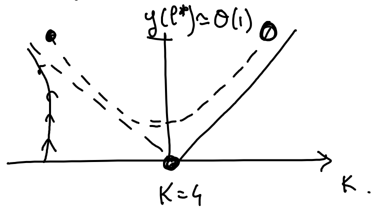
$\xi(\ell=0)$		$\xi(\ell^*)$	
$y(\ell=0)$	$\longrightarrow$	$y(\ell^*) \simeq 1$	fixes ℓ^*
$K(\ell=0)$		$K(\ell^*)$	

$\Downarrow$ know how to compute $\xi(\ell^*)$

[expanding the cosine $\quad \xi(\ell^*) = \dfrac{1}{\sqrt{y(\ell^*)}} \approx \mathcal{O}(1)$

$$\xi(\ell=0) = e^{\ell^*} \xi(\ell^*)$$

$$a(\ell=0) = a_0 \quad \longrightarrow \quad a(\ell^*) = a_0 e^{\ell^*}$$



$$\frac{\partial K}{\partial \ell} = -y^2 K^2$$

$$\frac{\partial y}{\partial \ell} = \frac{1}{2}(K-4)\,y$$

Far from the transition. $\quad K \approx$ Cste

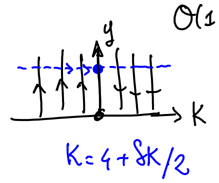
$$\frac{\partial y}{\partial \ell} = \frac{1}{2}(4-K)\,y \quad \Rightarrow \quad \mathrm{Log}\left(\frac{y(\ell)}{y(0)}\right) = \frac{1}{2}(4-K)\,\ell .$$

$$\mathrm{Log}\left(\frac{1}{y(0)}\right) = \frac{1}{2}(4-K)\,\ell^*$$

$$\ell^* \quad \frac{2}{4-K}\,\mathrm{Log}\left(\frac{1}{y(0)}\right)$$

$$\xi(\ell=0) = e^{\ell^*}\,\xi(\ell^*) \equiv e^{\frac{2}{4-K}\log\left(\frac{1}{y(0)}\right)}\,\underbrace{\xi(\ell^*)}_{\mathcal{O}(1)}$$

$$= \left(\frac{1}{y(0)}\right)^{\frac{2}{4-K}}\,\text{Cste}$$



⚠ incorrect flow.

$$\xi(\ell=0) \equiv e^{\frac{4}{\delta K}\log\left(\frac{1}{y(0)}\right)}\,\text{Cste} = \text{Cste}\; e^{\frac{A}{K-K_{cr}}}$$

$$\xi_{true}(\ell=0) \equiv e^{\frac{A}{\sqrt{K-K^*}}}$$ exponen

Superconductors / Superfluid.

$\psi(x)$ order parameter $\qquad \psi(x) = \psi_0\, e^{i\theta(x)}$

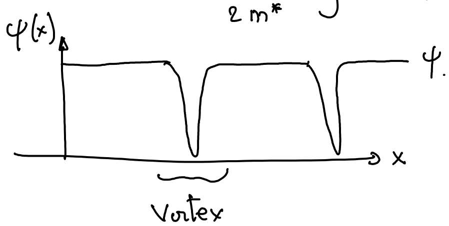
$$F[\psi,\psi^*] = \int d^d r \left[\alpha\,|\psi|^2 + \beta\left(|\psi|^2\right)^2\right] + \frac{1}{2m^*}(\nabla\psi)^*(\nabla\psi)$$

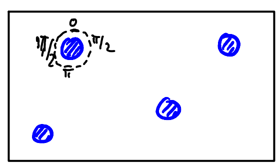
$T \ll T_c \qquad \alpha \ll 0$

$\psi_0 \qquad -|\alpha|\,\psi_0^2 + \beta\,\psi_0^4 \quad \rightarrow$ fixes the amplitude ψ_0

$$F_{phase} = \int d^d r\,\frac{\psi_0^2}{2m}\left(\nabla e^{i\theta}\right)^*\left(\nabla e^{i\theta}\right) = \frac{\psi_0^2}{2m^*}\int d^d r\,(\nabla\theta)^2$$

$$= \frac{\rho_s}{2m^*}\int d^d r\,(\nabla\theta)^2$$



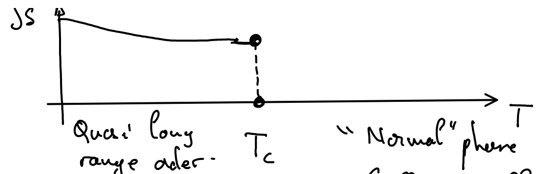


Superfluids / Superconductors $\Rightarrow$ xy Hamiltonians.

2 dimensions $\Rightarrow$ Superfluid or Superconducting film.

He_4

$K^* \rightarrow$ directly related to the superfluid density ρ_s.



(all superfluid correlations decrease exponentially)

$$\bar{F} = \frac{\rho_s^*}{2m^*}\int(\nabla\theta)^2$$

Universal jump of the superfluid density