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Bosons and Fermions

I.] Basis of second quantization:

N bosons or Fermions

$$H = \sum_{i=1}^{N} \frac{P_i^2}{2m} + \cdots$$

$$|\psi\rangle =$$

distinguishable particles.

$|\alpha)$ complete basis for 1 particle.

$$|\alpha_1, \alpha_2, \ldots \alpha_N) = |\alpha_1) \otimes |\alpha_2) \otimes | \ldots \otimes |\alpha_N)$$

- Particles are indistinguishable.

$|\alpha_i)$ Complete basis of 1 particle state.

$|\alpha_1) \ldots |\alpha_\Omega)$ Ω : typically volume of the system.

$|\alpha_1) \longrightarrow n_1$ number of particles in state $|\alpha_1)$

$\vdots$

$|\alpha_\Omega) \longrightarrow n_\Omega$.

$|n_1, n_2, n_3, \ldots, n_\Omega\rangle$ (+ basis $|\alpha_1) \ldots |\alpha_\Omega)$)

$$\langle r_1, r_2, r_3, \ldots \underset{\substack{\uparrow \\ n_1+n_2+n_3+\cdots n_\Omega}}{r_N} | n_1 \ldots n_\Omega\rangle = \frac{\sqrt{N!}}{\sqrt{n_1!} \ldots \sqrt{n_\Omega!}} S_\pm [\alpha_1 \ldots \alpha_\Omega]$$

$$S_\pm = \frac{1}{N!} \sum_P (\pm 1)^{S(P)} \underbrace{\alpha_1(r_{P(1)})\,\alpha_1(r_{P(2)}) \ldots}_{n_1} \underbrace{\alpha_2(r_{P(\ldots)}) \ldots}_{n_2} \ldots \underbrace{\alpha_\Omega(r_{P(\ldots)})}_{n_\Omega}$$

$|n_1, \dots n_\Omega\rangle$ any state with # of particles ranging from $0 \to \infty$

Fock space. $\mathcal{F} = H_0 \oplus H_1 \oplus H_2 \oplus H_3 \cdots$

$$\langle n_1, \dots n_\Omega | n_1', \dots n_\Omega' \rangle = \delta_{n_1 n_1'} \dots \delta_{n_\Omega n_\Omega'}.$$

bosons

$$\begin{cases} a_i^\dagger \, |n_1, n_2, \dots n_i, \dots n_\Omega\rangle = \sqrt{n_i + 1} \, |n_1, \dots, n_i + 1, \dots n_\Omega\rangle \\ a_i \, |n_1, n_2, \dots, n_i \dots n_\Omega\rangle = \sqrt{n_i} \, |n_1, \dots, n_i - 1, \; n_\Omega\rangle \end{cases}$$

$$|n_1, n_2 \dots n_\Omega\rangle = \frac{(a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} \dots (a_\Omega^\dagger)^{n_\Omega}}{\sqrt{n_1!}\sqrt{n_2!} \dots \sqrt{n_\Omega!}} \underbrace{|0, 0, 0, 0, \dots 0\rangle}_{|\phi\rangle}$$

$$\begin{cases} [a_i, a_j] = 0 \qquad [a_i^\dagger, a_j^\dagger] = 0 \\ [a_i, a_j^\dagger] = \delta_{ij} \end{cases}$$

- Fully described.
  - Complete basis of 1 particle states $|\alpha_i)$
  - A vacuum state $|\phi\rangle$ $[\forall i \;\; a_i|\phi\rangle = 0]$
  - a set of operators a_i, $a_i^\dagger$

    $[a_i, a_j] = 0 \qquad [a_i, a_j^\dagger] = \delta_{ij}$ (bosons)

Canonical Commutation rules.

Fermions: $\{a_i, a_j\} = 0 \qquad \{a_i, a_j^\dagger\} = \delta_{ij}$

$$\{a, b\} = ab + ba.$$

$$\{c_i^\dagger, c_i^\dagger\} = 2\, c_i^\dagger c_i^\dagger = 0$$

Express the operators in the second quantization basis.

- One body operator

ex: $H = \sum_{i=1}^{N} H_i = \sum_{i=1}^{N} \frac{p_i^2}{2m}$.

$$\mathcal{O} = \sum_{i=1}^{N} \mathcal{O}_i^{(1)}$$

$$\mathcal{O} = \sum_{\alpha,\beta} (\alpha|\mathcal{O}^{(1)}|\beta)\, c_\alpha^\dagger c_\beta$$

α, β are vectors of a complete basis of 1 particle states

- Complete basis of $|k\rangle$ states

$$\langle r|k\rangle = \frac{1}{\sqrt{\Omega}} e^{ikr}.$$

- Density operator

$$\langle \rho(r_0)\rangle = |\psi(r_0)|^2. \qquad \langle \mathcal{O}\rangle = \langle\psi|\mathcal{O}|\psi\rangle$$

$$R|r_0\rangle = r_0|r_0\rangle \qquad \rho(r_0) = |r_0\rangle\langle r_0|$$

Density operator. $\rho(r_0) = \sum_i \left[|r_0\rangle\langle r_0|\right]_{i\text{th}}.$

Complete basis $|r)$ basis of positions.

$$\mathcal{O} = \sum_{r,r'} (r|\mathcal{O}^{(1)}|r')\, c_r^\dagger c_{r'}$$

$$= \sum_{r r'} (r|r_0)(r_0|r')\, c_r^\dagger c_{r'} = c_{r_0}^\dagger c_{r_0}$$

$$\rho(r_0) = c_{r_0}^\dagger c_{r_0}.$$

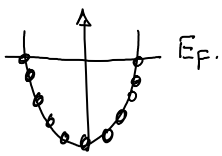
Complete basis $|k)$ basis of momenta

Complete basis $|k)$ basis of momenta

$$\mathcal{O} = \sum_{k_1 k_2} (k_1 | r_0)(r_0 | k_2) \quad c_{k_1}^{+} c_{k_2}$$

$$\mathcal{O} = \frac{1}{\Omega} \sum_{k_1 k_2} e^{i(k_2 - k_1) r_0} \quad c_{k_1}^{+} c_{k_2}$$

Free electrons.



$$|\psi\rangle = c_{k_F}^{+} \cdots c_{k_4}^{+} c_{k_3}^{+} c_{k_2}^{+} c_{k_1}^{+} c_{k_0}^{+} |\phi\rangle$$

$$|F\rangle = \prod_{k,\ \xi_k < E_F} c_k^{+} \quad |\phi\rangle$$

$$\langle F | \rho(r_0) | F \rangle = \underbrace{\langle \phi | (c_{k_0} c_{k_1} \cdots c_{k_F})}_{\langle F |} \frac{1}{\Omega} \sum_{k_1 k_2} e^{i(k_2 - k_1) r_0} c_{k_1}^{+} c_{k_2} \quad (c_{k_F}^{+} \cdots\cdots c_{k_0}^{+}) |\phi\rangle$$

Example: $\quad |F\rangle = c_{k_1}^{+} |\phi\rangle$

$$\langle F | \frac{1}{\Omega} \sum_{kk'} e^{i(k'-k) r_0} \quad c_k^{+} c_{k'} | F \rangle$$

$$= \frac{1}{\Omega} \sum_{kk'} e^{i(k'-k) r_0} \qquad \langle \phi | c_{k_1} \, c_k^{+} c_{k'} \, c_{k_1}^{+} | \phi \rangle$$

Fermions. $\quad c_{k'} c_{k_1}^{+} + c_{k_1}^{+} c_{k'} = \delta_{k_1 k'}$

$$c_{k'} c_{k_1}^{+} = \delta_{kk'} - c_{k_1}^{+} c_{k'}$$

$$\langle \phi | c_{k_1} c_k^{+} [\delta_{kk'} - c_{k_1}^{+} c_{k'}] | \phi \rangle$$

$$= \langle \phi | c_{k_1} c_k^{+} | \phi \rangle \delta_{kk'} - \underbrace{\langle \phi | c_{k_1} c_k^{+} c_{k_1}^{+} c_{k'} | \phi \rangle}_{0}$$

$$= \langle \phi | c_{k_1} c_k^{+} | \phi \rangle \delta_{kk'}$$

$$= \langle \phi | (\delta_{k_1 k} - c_k^\dagger c_{k_1}) | \phi \rangle \, \delta_{kk'}$$

$$= \langle \phi | \phi \rangle \, \delta_{k_1 k} \, \delta_{kk'} \quad = \delta_{k_1 k} \, \delta_{kk'}$$

$$\langle F | \rho(r_0) | F \rangle = \frac{1}{\Omega} \sum_{kk'} e^{i(k'-k) r_0} \, \delta_{k_1 k} \, \delta_{kk'}$$

$$= \frac{1}{\Omega} \sum_{k} \delta_{k_1 k} = \frac{1}{\Omega}$$

$$| \psi_{BCS} \rangle = \prod_k \Big(\underbrace{u_k}_{\text{0 part.}} + \underbrace{v_k \, c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger}_{\text{2 particles.}} \Big) | \phi \rangle$$

Coh. Sup. : 0, 2, 4, 6, ... ∞ particles

Exercice :

$$\langle \psi_{BCS} | \psi_{BCS} \rangle = 1.$$

Hamiltonian of Free particles.

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m}.$$

$$H = \sum_{k_1, k_2} (k_1 | \frac{p^2}{2m} | k_2) \, c_{k_1}^\dagger c_{k_2}$$

$$= \sum_{k_1 k_2} \frac{\hbar^2 k_2^2}{2m} (k_1 | k_2) \, c_{k_1}^\dagger c_{k_2}$$

$$H = \sum_{k} \frac{\hbar^2 k^2}{2m} c_k^\dagger c_k.$$

$c_\alpha^\dagger c_\alpha$: counting the particles with quantum number α.

Exercise $\quad \langle F | H | F \rangle$

$\langle \psi_{BCS} | H | \psi_{BCS} \rangle$ (start with 1 value of k)

$$H - \mu N = \sum_{k} \epsilon_k \, c_k^\dagger c_k - \mu \sum_k c_k^\dagger c_k$$

$$= \sum_k (\epsilon_k - \mu) \, c_k^\dagger c_k$$

$$Z = \mathrm{Tr}[e^{-\beta H}]$$

$$\langle n_k \rangle \qquad \tilde{H} = H - \mu N = \sum_k \overbrace{(\epsilon_k - \mu)}^{\xi_k} c_k^\dagger c_k$$

$$\langle n_k \rangle = \frac{1}{Z} \mathrm{Tr}[e^{-\beta \tilde{H}} n_k] \qquad Z = \mathrm{Tr}[e^{-\beta \tilde{H}}]$$

$$\mathrm{Tr}[\, e^{-\beta \sum_k \xi_k c_k^\dagger c_k} \,]$$

Complete basis $|n_1, n_2, \ldots, n_\Omega\rangle \rightarrow |n_{k_1}, n_{k_2}, \ldots, n_{k_\Omega}\rangle$

$$[c_{k_1}^\dagger c_{k_1}, c_{k_2}^\dagger c_{k_2}] = \qquad\qquad k_1 = k_2 \Rightarrow 0$$

$$c_{k_1}^\dagger [c_{k_1}, c_{k_2}^\dagger c_{k_2}] + [c_{k_1}^\dagger, c_{k_2}^\dagger c_{k_2}] c_{k_1}$$

<u>fermions</u> $[A, BC] = ABC - BCA = ABC + BAC - BAC - BCA$

$$= \{AB\}C - B\{A, C\}$$

$$[c_{k_1}^\dagger c_{k_1}, c_{k_2}^\dagger c_{k_2}] = 0$$

<u>bosons</u>

$$[A, BC] = ABC - BCA = ABC - BAC + BAC - BCA$$

$$= [A, B]C + B[A, C]$$

$$[c_{k_1}^\dagger c_{k_1}, c_{k_2}^\dagger c_{k_2}] = 0$$

$$[c_{k_1}^\dagger c_{k_1},\ c_{k_2}^\dagger c_{k_2}] = 0$$

$$e^{-\beta \sum_k \xi_k c_k^\dagger c_k} = \prod_k \left(e^{-\beta \xi_k c_k^\dagger c_k} \right)$$

$$Z = \sum_{n_1 \dots n_\Omega} \langle n_{k_1} \dots n_{k_\Omega} | \prod_k \left(e^{-\beta \xi_k c_k^\dagger c_k} \right) | n_{k_1} \dots n_{k_\Omega} \rangle$$

$$\left(e^{-\beta \xi_{k_1} c_{k_1}^\dagger c_{k_1}} \right) \left(e^{-\beta \xi_{k_2} c_{k_2}^\dagger c_{k_2}} \right) \dots \left(\quad \right)_\Omega$$

$$Z = \sum_{n_1 \dots n_\Omega} \prod_k e^{-\beta \xi_k n_k} \underbrace{\langle n_1 \dots n_\Omega | n_1 \dots n_\Omega \rangle}_{1}$$

$$Z = \prod_k \left(\sum_{n_k} e^{-\beta \xi_k n_k} \right)$$

$$\mathrm{Tr}\left[e^{-\beta \tilde{H}} c_{k_0}^\dagger c_{k_0} \right] = \sum_{n_1 \dots n_\Omega} \langle n_1 n_2 \dots n_\Omega | e^{-\beta \tilde{H}} c_{k_0}^\dagger c_{k_0} | n_1 \dots n_{\Omega} \rangle$$

$$= \left[\prod_{k \neq k_0} \left(\sum_{n_k} e^{-\beta \xi_k n_k} \right) \right] \left(\sum_{n_{k_0}} e^{-\beta \xi_{k_0} n_{k_0}} n_{k_0} \right)$$

$$\langle c_{k_0}^\dagger c_{k_0} \rangle = \frac{\sum_{n_{k_0}} e^{-\beta \xi_{k_0} n_{k_0}} n_{k_0}}{\sum_{n_{k_0}} e^{-\beta \xi_{k_0} n_{k_0}}}$$

- Fermions:

$$\langle c_{k_0}^\dagger c_{k_0} \rangle = \frac{0 + e^{-\beta \xi_{k_0}}}{1 + e^{-\beta \xi_{k_0}}}$$

$$\langle n_{k_0} \rangle = \frac{1}{e^{\beta \xi_{k_0}} + 1}$$

- bosons:

$$\langle b_{k_0}^\dagger b_{k_0} \rangle = \frac{\sum_{n=0}^{+\infty} e^{-\beta \xi_{k_0} n}\, n}{\sum^{+\infty} \dots}$$

$$\overline{\sum_{n=0}^{+\infty} e^{-\beta \xi_{h_0} n}}$$

$$\frac{\partial}{\partial u} \sum_{n=0}^{+\infty} e^{-un} = \sum_{n=0}^{+\infty} (-n)\, e^{-un}$$

$$-\frac{\partial}{\partial u} \operatorname{Log}\left[\sum_{n=0}^{+\infty} e^{-un}\right] = \frac{1}{\sum e^{-un}} \sum n\, e^{-un}$$

$$\sum_{n=0}^{+\infty} e^{-un} = \sum_{n=0}^{+\infty} \left(e^{-u}\right)^{n} = \frac{1}{1-e^{-u}}$$

$$\to \operatorname{Log}\left[1-e^{-u}\right]$$

$$\frac{\partial}{\partial u}\left(\operatorname{Log}\left[1-e^{-u}\right]\right) = \frac{e^{-u}}{1-e^{-u}} = \frac{1}{e^{u}-1}$$

$$\langle b^{+}_{h_0} b_{h_0} \rangle = \frac{1}{e^{\beta \xi_{h_0}} - 1}$$

Exercise.

$$\langle \Psi_{BCS} | N | \Psi_{BCS} \rangle = \bar{N}$$

$$\langle \Psi_{BCS} | (N-\bar{N})^2 | \Psi_{BCS} \rangle = \delta N^2$$

$\frac{\delta N}{\bar{N}} \to 0$ in the thermodynamic limit

Two body operators

operator that involves. two particles (pair of particles)

$$\mathcal{O} = \frac{1}{2} \sum_{i,j=1}^{N} \mathcal{O}^{(2)}_{i,j}$$

Coulomb potential. $V(r) = \frac{e^2}{4\pi \varepsilon_0 r}$

$V(r_1 - r_2) \quad \leftarrow$ classical particles.

$\hat{R}$: measuring the position of a particle

$R \,|\, r_0) = r_0 \,|\, r_0)$.

$$V(\hat{R}_1 - \hat{R}_2) \qquad V(\hat{R} \otimes \mathbb{1} - \mathbb{1} \otimes \hat{R})$$

$$|\psi\rangle \qquad \langle r_1\, r_2 | \psi\rangle = \psi(r_1, r_2)$$

$$V = \frac{1}{2} \sum_{i,j} V(\hat{R}_i - \hat{R}_j)$$

2 particles : complete basis. of two particle state $|A\rangle$ $|B\rangle$

$$V = \sum_{A,B} |A\rangle\langle A| V | B\rangle\langle B|$$

$|A\rangle = |\alpha) \otimes |\beta)$ (distinguishable particles).

$|\alpha)$ $|\beta)$ are two state of the 1 particle basis.

$$|A\rangle = \frac{1}{\sqrt{2}} \left[|\alpha) \otimes |\beta) \pm |\beta) \otimes |\alpha) \right]$$

$$|A\rangle \longrightarrow c^{+}_{\alpha} c^{+}_{\beta} |\phi\rangle$$

$$|B\rangle \longrightarrow c^{+}_{\gamma} c^{+}_{\delta} |\phi\rangle$$

$$V = \sum_{\substack{\alpha\beta \\ \gamma\delta}} \underset{(\alpha\beta| \quad |\gamma\delta)}{\langle A | V | B\rangle} \overset{|A\rangle\langle B|}{c^{+}_{\alpha} c^{+}_{\beta} |\phi\rangle\langle\phi| c_{\delta} c_{\gamma}}$$

$$\boxed{V = \frac{1}{2} \sum_{\substack{\alpha\beta \\ \gamma\delta}} \langle \alpha\beta | V | \gamma\delta \rangle \; c^\dagger_\alpha c^\dagger_\beta c_\delta c_\gamma.}$$

ordered wave function

$|\alpha\rangle \otimes |\beta\rangle$
① ②
$|\gamma\rangle \otimes |\delta\rangle$

• Position basis.

$$V = \frac{1}{2} \sum_{\substack{r_1 r_2 \\ r_3 r_4}} (r_1 r_2 | V^{(2)} | r_3 r_4) \; c^\dagger_{r_1} c^\dagger_{r_2} c_{r_4} c_{r_3}$$

$$V^{(2)} = V_0(\hat{R}_1 - \hat{R}_2)$$

$$(r_1 r_2 | V_0(\hat{R}_1 - \hat{R}_2) | r_3 r_4) = (r_1 r_2 | r_3 r_4) \, V_0(r_3 - r_4)$$

$$= \delta_{r_1 r_3} \delta_{r_2 r_4} V_0(r_3 - r_4)$$

$cc^\dagger - c^\dagger c = \delta$

$$V = \frac{1}{2} \sum_{r_1 r_2} V_0(r_1 - r_2) \; c^\dagger_{r_1} c^\dagger_{r_2} c_{r_2} c_{r_1}.$$

bosons.
$$c^\dagger_{r_1} c^\dagger_{r_2} c_{r_2} c_{r_1} = c^\dagger_{r_1} c^\dagger_{r_2} c_{r_1} c_{r_2}$$
$$= c^\dagger_{r_1} \left[c_{r_1} c^\dagger_{r_2} - \delta_{r_1 r_2} \right] c_{r_2}$$
$$= c^\dagger_{r_1} c_{r_1} c^\dagger_{r_2} c_{r_2} - \delta_{r_1 r_2} c^\dagger_{r_1} c_{r_2}$$

fermions
$$c^\dagger_{r_1} c^\dagger_{r_2} c_{r_2} c_{r_1} = - c^\dagger_{r_1} c^\dagger_{r_2} c_{r_1} c_{r_2}$$
$$= - c^\dagger_{r_1} \left[\delta_{r_1 r_2} - c_{r_1} c^\dagger_{r_2} \right] c_{r_2}$$
$$= c^\dagger_{r_1} c_{r_1} c^\dagger_{r_2} c_{r_2} - \delta_{r_1 r_2} c^\dagger_{r_1} c_{r_2}$$

$$V = \frac{1}{2} \sum_{r_1 r_2} V_0(r_1 - r_2) \; c^\dagger_{r_1} c_{r_1} c^\dagger_{r_2} c_{r_2}$$

$$- \frac{1}{2} \sum_{r_1} V(r=0)\, c^\dagger_{r_1} c_{r_1}. \qquad \leftarrow \frac{1}{2} V(r=0)\, N_{tot}.$$

$$c^\dagger_{r_1} c_{r_1} = \rho(r_1) \qquad V = \frac{1}{2} \sum_{r_1 r_2} V(r_1 - r_2)\, \rho(r_1)\, \rho(r_2)$$

* Momentum basis.

$$V = \frac{1}{2} \sum_{\substack{k_1 k_2 \\ k_3 k_4}} (k_1 k_2 | V | k_3 k_4)\, c^\dagger_{k_1} c^\dagger_{k_2} c_{k_4} c_{k_3}$$

$$(k_1 k_2 | V | k_3 k_4) = \int dr_1 dr_2\, (k_1 k_2 | V | r_1 r_2)(r_1 r_2 | k_3 k_4)$$

$$V_0(\hat{R}_1 - \hat{R}_2)$$

$$= \int dr_1 dr_2\, V_0(r_1 - r_2)\, (k_1 k_2 | r_1 r_2)(r_1 r_2 | k_3 k_4)$$

$$= \frac{1}{\Omega^2} \int dr_1 dr_2\, V_0(r_1 - r_2)\, e^{-i k_1 r_1} e^{-i k_2 r_2} e^{i k_3 r_1} e^{i k_4 r_2}$$

$$R = \frac{r_1 + r_2}{2} \qquad r = r_1 - r_2 \qquad r_1 = R + \frac{r}{2} \qquad r_2 = R_1 - \frac{r}{2}$$

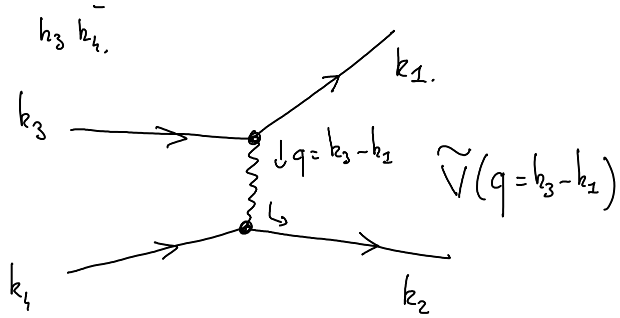
$$= \frac{1}{\Omega^2} \int dR\, dr\, V(r)\, e^{i[k_3 + k_4 - k_1 - k_2] R}\, e^{i[k_3 - k_4 - (k_1 - k_2)] \frac{r}{2}}$$

$$= \frac{1}{\Omega} \delta_{k_3 + k_4,\, k_1 + k_2} \int dr\, V_0(r)\, e^{i(k_3 - k_1) r}$$

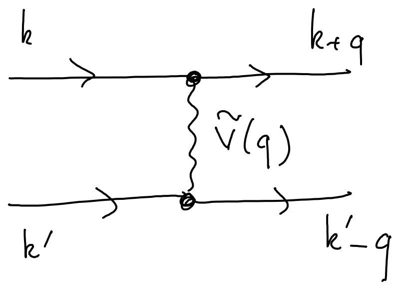
$$k_3 + k_4 = k_1 + k_2 \qquad k_3 - k_1 = k_2 - k_4$$

$$= \frac{1}{\Omega} \delta_{k_3 + k_4,\, k_1 + k_2}\, \tilde{V}_0(k_3 - k_1)$$

$$V = \frac{1}{2} \sum_{\substack{k_1 k_2 \\ k_3 k_4}} \delta_{k_3 + k_4,\, k_1 + k_2}\, \tilde{V}(k_3 - k_1)\, c^\dagger_{k_1} c^\dagger_{k_2} c_{k_4} c_{k_3}$$



$$k_3 + k_4 = k_1 + k_2$$



$$H = \sum_{k} \xi_k\, c_k^{+} c_k + \frac{1}{2} \sum_{\substack{k, k' \\ q}} c^{+}_{k+q}\, c^{+}_{k'-q}\, c_{k'}\, c_k\, \tilde{V}(q).$$

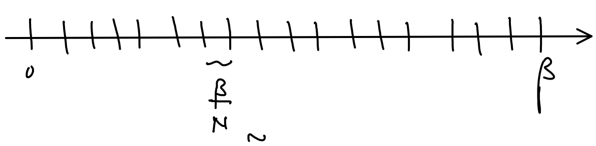
$$\langle O(t)\, O'(0) \rangle \qquad Z = \mathrm{Tr}\left[e^{-\beta \tilde{H}} \right]$$

$$\mathrm{Tr}\left[e^{-\beta H}\, e^{iHt}\, O\, e^{-iHt}\, O' \right]$$

Mahan : "Many particle Physics."

2.) Path integral for bosons.

$$Z = \mathrm{Tr}\left[e^{-\beta \tilde{H}} \right] \qquad e^{-\beta \tilde{H}} = \prod \left(e^{-\frac{\beta}{N} \tilde{H}} \right)$$



$$\langle \phi_{N+1} |\, e^{-\varepsilon \tilde{H}}\, | \phi_N \rangle \longrightarrow \text{Computable.}$$

Coherent states :

$$|\Phi\rangle \qquad a_\alpha |\Phi\rangle = \phi_\alpha |\Phi\rangle$$

$$|\Phi\rangle = e^{\sum_\alpha \phi_\alpha a^+_\alpha} |\phi\rangle$$

$$a_{\alpha_0} |\Phi\rangle = a_{\alpha_0} e^{\sum_{\alpha} \phi_\alpha a^+_\alpha} |\phi\rangle$$

1 state α

$$= a_{\alpha_0} \left(1 + \phi_{\alpha_0} a^+_{\alpha_0} + \frac{1}{2}\phi^2_{\alpha_0} a^{+2}_{\alpha_0} + \cdots\right) |\phi\rangle$$

$$= \phi_{\alpha_0} \underbrace{a_{\alpha_0} a^+_{\alpha_0}}_{1 + a^+_{\alpha_0} a_{\alpha_0}} |\phi\rangle + \frac{1}{2}\phi^2_{\alpha_0} a_{\alpha_0} a^{+2}_{\alpha_0} |\phi\rangle + \cdots$$

$$= \phi_{\alpha_0} |\phi\rangle + \cdots \frac{1}{n!} \phi^n_{\alpha_0} [a_{\alpha_0}, (a^+_{\alpha_0})^n] |\phi\rangle$$

$$a_{\alpha_0} (a^+_{\alpha_0})^n = (a^+_{\alpha_0})^n a_{\alpha_0} + [a_{\alpha_0}, (a^+_{\alpha_0})^n].$$

$$[a_{\alpha_0}, (a^+_{\alpha_0})^n] = n (a^+_{\alpha_0})^{n-1} [a_{\alpha_0}, a^+_{\alpha_0}] = n (a^+_{\alpha_0})^{n-1}$$

$$a_{\alpha_0} |\Phi\rangle = \phi_{\alpha_0} |\phi\rangle + \cdots \frac{1}{(n-1)!} \phi^n_{\alpha_0} (a^+_{\alpha_0})^{n-1} |\phi\rangle \cdots$$

$$= \phi_{\alpha_0} \left[|\phi\rangle + \frac{1}{(n-1)!} \phi^{n-1}_{\alpha_0} (a^+_{\alpha_0})^{n-1} |\phi\rangle + \cdots \right.$$

$$= \phi_{\alpha_0} e^{\phi_{\alpha_0} a^+_{\alpha_0}} |\phi\rangle$$

$$|\Phi\rangle = e^{\sum_\alpha \varphi_\alpha a^+_\alpha} |\phi\rangle$$

$$\begin{cases} a_{\alpha_0} |\Phi\rangle = \varphi_{\alpha_0} |\Phi\rangle \\ a^+_{\alpha_0} |\Phi\rangle = \dfrac{\partial}{\partial \varphi_{\alpha_0}} |\Phi\rangle \end{cases}$$

$$\langle \Phi_1 | \Phi_2 \rangle = ?$$

$$|\Phi_1\rangle = e^{\sum_\alpha \varphi_\alpha a^+_\alpha} |\phi\rangle$$

$$|\Phi_1\rangle = e^{\sum_\alpha \varphi_\alpha a^\dagger_\alpha} |\phi\rangle$$

$$|\Phi_2\rangle = \sum_{n_1 \dots n_\Omega} \varphi_{n_1 \dots n_\Omega} |n_1, n_2 \dots n_\Omega\rangle$$

$$\varphi_{n_1 \dots n_\Omega} = \frac{\varphi_{\alpha_1}^{n_1} \varphi_{\alpha_2}^{n_2} \cdots \varphi_{\alpha_\Omega}^{n_\Omega}}{\sqrt{n_1!}\sqrt{n_2!} \cdots \sqrt{n_\Omega!}}$$

$$\langle \Phi_1 | \Phi_2 \rangle = \sum_{\substack{n_1 - n_\Omega \\ n'_1 - n'_\Omega}} \left(\langle n_1 - n_\Omega | \frac{(\varphi^*_{1,\alpha_1})^{n_1}}{\sqrt{n_1!}} \cdots \right) \left(\frac{(\varphi_{2,\alpha_1})^{n'_1}}{\sqrt{n'_1!}} \cdots \right) |n'_1 \dots n'_\Omega\rangle$$

$$= \sum_{n_1 n_2 \dots n_\Omega} \frac{(\varphi^*_{1,\alpha_1} \varphi_{2,\alpha_1})^{n_1}}{n_1!} \frac{(\varphi^*_{1,\alpha_2} \varphi_{2,\alpha_2})^{n_2}}{n_2!} \cdots$$

$$= e^{\sum_\alpha \varphi^*_{1\alpha} \varphi_{2\alpha}}$$

<u>Closure relation:</u>

$$\prod_\alpha \int \frac{d\varphi^*_\alpha \, d\varphi_\alpha}{2i\pi} e^{-\sum_\alpha \varphi^*_\alpha \varphi_\alpha} |\Phi(\varphi)\rangle \langle \Phi(\varphi)| = \mathbb{1}$$

$$[a_{\alpha_0}, |\Phi\rangle\langle\Phi|] = a_{\alpha_0} |\Phi\rangle\langle\Phi| - |\Phi\rangle\langle\Phi| a_{\alpha_0}$$

$$= \varphi_{\alpha_0} |\Phi\rangle\langle\Phi| - |\Phi\rangle\langle\Phi| a_{\alpha_0}.$$

$$a^\dagger_{\alpha_0} |\Phi\rangle = \frac{\partial}{\partial \varphi_{\alpha_0}} |\Phi\rangle \longrightarrow \langle\Phi| a_{\alpha_0} = \frac{\partial}{\partial \varphi^*_{\alpha_0}} \langle\Phi|$$

$$[a_{\alpha_0}, |\Phi\rangle\langle\Phi|] = \left(\varphi_{\alpha_0} - \frac{\partial}{\partial \varphi^*_{\alpha_0}}\right) |\Phi\rangle\langle\Phi|$$

$$\left[a_{\alpha_0} , \int \prod_\alpha \frac{d\varphi^* d\varphi}{2i\pi} e^{-\sum_\alpha \varphi^*_\alpha \varphi_\alpha} |\Phi\rangle\langle\Phi| \right]$$

$$= \int \prod_\alpha \frac{d\varphi^*_\alpha d\varphi_\alpha}{2i\pi} e^{-\sum_\alpha \varphi^*_\alpha \varphi_\alpha} \left(\varphi_{\alpha_0} - \frac{\partial}{\partial \varphi^*_{\alpha_0}} \right) |\Phi\rangle\langle\Phi|$$

integration by part.

$$\frac{\partial}{\partial \varphi^*_{\alpha_0}} e^{-\sum_\alpha \varphi^*_\alpha \varphi_\alpha} = -\varphi_{\alpha_0}.$$

$$\left[a_{\alpha_0} , \int \cdots\cdots \right] = 0 \qquad \left[a^+_{\alpha_0} , \int \cdots\cdots \right] = 0$$

Schur's theorem. $\left[a_\alpha , A \right] = \left[a^+_\alpha , A \right] = 0 \Rightarrow A \propto \mathbb{1}$

$$\langle \phi | \int \cdots |\Phi\rangle\langle\Phi| \, |\phi\rangle = 1.$$

$$\boxed{\int \prod_\alpha \frac{d\varphi^*_\alpha d\varphi_\alpha}{2i\pi} e^{-\sum_\alpha \varphi^*_\alpha \varphi_\alpha} |\Phi\rangle\langle\Phi| = \mathbb{1}}$$

- Path integral:

$$H = \sum_k \xi_k b^+_k b_k + \frac{1}{2} \sum_{k_1 k_2 q} \tilde{V}(q) \, b^+_{k_1+q} b^+_{k_2-q} b_{k_2} b_{k_1}.$$

write the b operators on the right b^+ on the left.

[Normal ordered form]

$$Z = \mathrm{Tr}[A] = \sum_n \langle n | A | n \rangle$$

$$= \sum_n \int \prod_\alpha \frac{d\varphi^*_\alpha d\varphi_\alpha}{2i\pi} e^{-\sum_\alpha \varphi^*_\alpha \varphi_\alpha} \langle n | A | \Phi \rangle \langle \Phi | n \rangle$$

$$\sum_n \langle \Phi | n \rangle \langle n | A | \Phi \rangle = \langle \Phi | A | \Phi \rangle$$

$$\mathrm{Tr}[A] = \int \prod_\alpha \frac{d\varphi_\alpha^* d\varphi_\alpha}{2i\pi} e^{-\sum_\alpha \varphi_\alpha^* \varphi_\alpha} \langle \Phi | A | \Phi \rangle$$

$$\mathrm{Tr}[e^{-\beta \tilde{H}}] = \mathrm{Tr}[e^{-\varepsilon \tilde{H}} e^{-\varepsilon \tilde{H}} e^{-\varepsilon \tilde{H}} \cdots\cdots e^{-\varepsilon \tilde{H}}]$$

$$= \int \prod_\alpha \frac{d\varphi_\alpha^* d\varphi}{2i\pi} e^{-\sum_\alpha \varphi_\alpha^* \varphi_\alpha} \langle \Phi | e^{-\varepsilon H} \wedge e^{-\varepsilon H} \wedge \cdots\cdot \wedge e^{-\varepsilon \tilde{H}} \wedge | \Phi \rangle$$

$$\int \prod_\alpha \frac{d\varphi_\alpha^* d\varphi_\alpha}{2i\pi} e^{-\sum_\alpha \varphi_\alpha^* \varphi_\alpha} \langle \Phi | e^{-\varepsilon \tilde{H}} | \Phi_1 \rangle \wedge \langle \Phi_1 | e^{-\varepsilon \hat{H}} | \Phi \rangle$$

$$\prod_\alpha \frac{d\varphi_{\alpha_1}^* d\varphi_{\alpha_1}}{2i\pi} e^{-\sum \varphi_{\alpha_1}^* \varphi_{\alpha_1}}$$

$$\Big(\sum_\varphi \sum_{\varphi_1} \langle \varphi | e^{-\varepsilon \tilde{H}} | \varphi_1 \rangle \langle \varphi_1 | e^{-\varepsilon \tilde{H}} | \varphi \rangle$$

$$\langle \Phi_{n+1} | e^{-\varepsilon \tilde{H}} | \Phi_n \rangle \simeq \langle \Phi_{n+1} | (1 - \varepsilon \tilde{H}) | \Phi_n \rangle$$

$$b^+ b^+ b^+ \; b\, b\, b$$

replace $b_\alpha \to \varphi_{\alpha,n}$

$b_\alpha^+ \to \varphi_{\alpha,n+1}^*$

$$\langle \Phi_{n+1} | e^{-\varepsilon \tilde{H}} | \Phi \rangle \to \langle \Phi_{n+1} | \Phi_n \rangle e^{-\varepsilon \cdot \tilde{H}[b_\alpha^+ \to \varphi_{\alpha,n+1}^*,\, b_\alpha \to \varphi_{\alpha n}]}$$

$$= e^{\sum_\alpha \varphi_{\alpha,n+1}^* \varphi_{\alpha,n}} \; e^{-\varepsilon \tilde{H}[\varphi_{\alpha,n+1}^*, \varphi_{\alpha,n}]}$$

$$Z = \int \prod_\alpha \frac{d\varphi_\alpha^* d\varphi_\alpha}{(2i\pi)} \Big(\prod_\alpha \frac{d\varphi_{\alpha,1}^* d\varphi_{\alpha,1}}{2i\pi} \cdots\cdots \Big) \prod_j e^{-\sum_\alpha \varphi_{\alpha,j}^* \varphi_{\alpha,j}}$$

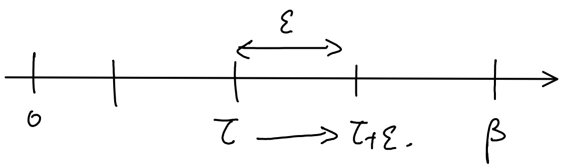
$$\prod e^{\sum_\alpha \varphi_{\alpha,j+1}^* \varphi_{\alpha,j}} \; e^{-\varepsilon \tilde{H}[\varphi_{\alpha,j+1}^*, \varphi_{\alpha,j}]}$$

$$Z = \int \prod_\alpha \frac{d\varphi_\alpha^* d\varphi_\alpha}{(2i\pi)} \left(\prod_\alpha \frac{d\varphi_{\alpha,1}^* \, d\varphi_{\alpha,1}}{2i\pi} \cdots\cdots \right) \prod_j e^{-\sum_\alpha \varphi_{\alpha,j}^* \varphi_{\alpha,j}}$$

$$\prod_j \; e^{\sum_\alpha \varphi_{\alpha,j+1}^* \varphi_{\alpha,j}} \; e^{-\varepsilon \tilde{H}[\varphi_{\alpha,j+1}^*, \varphi_{\alpha,j}]}$$

$$Z = \int \left[\prod d\varphi^* d\varphi \;\; \cdots\cdots \right] \prod_j \; e^{\sum_\alpha (\varphi_{\alpha,j+1}^* - \varphi_{\alpha,j}^*)\varphi_{\alpha,j}}$$

$$e^{-\varepsilon \tilde{H}[\varphi_{\alpha,j+1}^*, \varphi_{\alpha,j}]}$$



$$\varphi_\alpha^*(\tau+\varepsilon) - \varphi_\alpha^*(\tau) = \varepsilon\, \partial_\tau \varphi^*(\tau)$$

$$\prod \; e^{\sum_\alpha \varepsilon \partial_\tau \varphi_\alpha^*(\tau)\varphi_\alpha(\tau) \;-\; \varepsilon H[\varphi_{\tau+\varepsilon}^*, \varphi_\tau]}$$

$$\simeq \; e^{\int_0^\beta d\tau \; \sum_\alpha \partial_\tau \varphi_\alpha^*(\tau)\, \varphi_\alpha(\tau) \;-\; \tilde{H}[\varphi^*(\tau), \varphi(\tau)]}$$

$$Z = \underbrace{\int}_{\text{periodic in } \tau} \mathcal{D}\varphi_\alpha^*(\tau)\, \mathcal{D}\varphi_\alpha(\tau) \;\; e^{\int_0^\beta d\tau \left[\sum_\alpha \partial_\tau \varphi_\alpha^*(\tau)\, \varphi_\alpha(\tau) - \tilde{H}[\varphi_\alpha^*(\tau), \varphi_\alpha(\tau)] \right]}$$

Free bosons. $\quad \tilde{H} = \sum_k \xi_k \, b_k^\dagger b_k .$

$$Z = \int \mathcal{D}\varphi_k^* \mathcal{D}\varphi_k \;\; e^{\sum_k \int_0^\beta d\tau \left[\partial_\tau \varphi_k^*(\tau) \varphi_k(\tau) \;-\; \xi_k \, \varphi_k^*(\tau) \varphi_k(\tau) \right]}.$$

$$\varphi(\tau) = \frac{1}{\beta} \sum_{\omega_n} e^{-i\omega_n \tau} \varphi(\omega_n) \qquad \omega_n = \frac{2\pi}{\beta} n .$$

periodicity in τ of $\varphi(\tau)$

$$\partial_\tau \varphi(\tau) = \frac{1}{\beta} \sum_{\omega_n} -i\omega_n \; e^{-i\omega_n \tau} \; \varphi(\omega_n)$$

$$\partial_\tau \varphi(\tau) = \frac{1}{\beta} \sum_{\omega_n} -i\omega_n \, e^{i\omega_n \tau} \, \varphi(\omega_n)$$

$$\int_0^\beta d\tau \; \frac{1}{\beta^2} \sum_{\substack{\omega_1 \\ \omega_2}} e^{+i\omega_1 \tau} (+i\omega_1) \, \varphi^*(\omega_1) \, e^{-i\omega_2 \tau} \varphi(\omega_2)$$

$$= \frac{1}{\beta} \sum_{\omega_1} (+i\omega_1) \, \varphi^*(\omega_1) \, \varphi(\omega_1)$$

$$\int_0^\beta d\tau \, \tilde{H}(\tau) = \sum_k \xi_k \int_0^\beta d\tau \; \varphi_k^*(\tau) \varphi_k(\tau)$$

$$= \frac{1}{\beta} \sum_{k \, \omega_n} \xi_k \; \varphi^*(\omega_n, k) \, \varphi(\omega_n, k).$$

$$\frac{1}{\beta} \sum_{\omega_n, k} (i\omega_n - \xi_k) \, \varphi^*_{k\omega_n} \varphi_{k,\omega_n}.$$

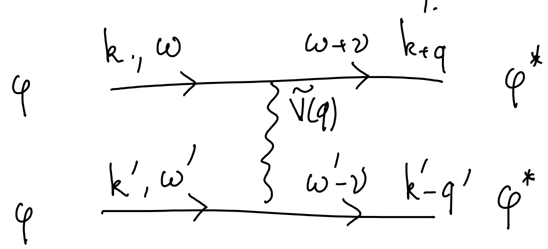
$$Z = \int \mathcal{D}\varphi^* \mathcal{D}\varphi \; e^{\frac{1}{\beta} \sum_{\omega_n k} (i\omega_n - \xi_k) \varphi^*_{k\omega_n} \varphi_{k,\omega_n}}$$

interactions-

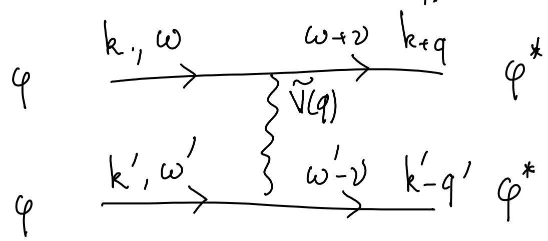
$$\int_0^\beta d\tau \sum_{\substack{k\, k' \\ q}} \tilde{V}(q) \; \varphi^*_{k+q}(\tau) \, \varphi^*_{k'-q}(\tau) \, \varphi_{k'}(\tau) \, \varphi_k(\tau)$$

$$= \frac{1}{\beta^4} \sum_{\substack{k\, k' \\ q}} \int_0^\beta d\tau \sum_{\substack{\omega_1 \omega_2 \\ \omega_3 \omega_4}} e^{i\omega_1 \tau} e^{i\omega_2 \tau} e^{-i\omega_3 \tau} e^{-i\omega_4 \tau} \; \varphi^*_{k+q,\omega_1} \, \varphi^*_{k'-q,\omega_2} \, \varphi_{k',\omega_3} \, \varphi_{k,\omega_4}$$

$$\frac{1}{\beta^3} \sum_{\substack{\omega_1 \omega_2 \\ \omega_3 \omega_4}} \delta_{\omega_1+\omega_2, \omega_3+\omega_4} \sum_{\substack{k\, k' \\ q}} \varphi^* \varphi^* \varphi \, \varphi$$



$\frac{}{\beta^3}\sum_{\substack{\omega_1\omega_2\\ \omega_3\omega_4}} \quad \delta_{\omega_1+\omega_2,\,\omega_3+\omega_4} \quad \sum_{\substack{kk'\\ q}}$



$$\frac{1}{2}\cdot \sum_{\substack{k,\omega,\, k',\omega'\\ q,\nu}} \tilde V(q)\,.\, \varphi^*_{k+q,\,\omega+\nu}\, \varphi^*_{k'-q,\,\omega'-\nu}\, \varphi_{k',\omega'}\, \varphi_{k,\omega}\,.$$

3) <u>Free bosons</u>

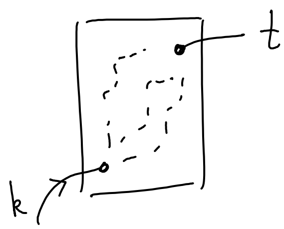
$$Z = \int \mathcal{D}\varphi^* \mathcal{D}\varphi \; e^{\int_0^\beta d\tau \sum_k \left[\partial_\tau \varphi^*_k \varphi_k - \xi_k \varphi^*_k(\tau)\varphi_k(\tau)\right]}$$

$$= \int \mathcal{D}\varphi^* \mathcal{D}\varphi \; e^{\sum_{k\omega_n} (i\omega_n - \xi_k)\, \varphi^*_{k\omega_n} \varphi_{k,\omega_n}}\,.$$

$$G(k,\tau) = -\left\langle T_\tau \; b_k(\tau)\; b_k^\dagger(0) \right\rangle = -\mathrm{Tr}\left[e^{-\beta \tilde H} e^{\tilde H \tau} b_k e^{-\tilde H\tau} b_k^\dagger\right]$$

$\downarrow$

$$G_{ret}(k,t) = -i\,\Theta(t) \left\langle \left[b_k(t), b_k^\dagger(0)\right]\right\rangle$$



$$G(k,\tau) = -\left\langle T_\tau \; b_k(\tau)\, b_k^\dagger(0)\right\rangle$$

$$\hookrightarrow -\frac{1}{Z}\int \mathcal{D}\varphi^* \mathcal{D}\varphi \quad \varphi_k(\tau)\,\varphi_k^*(0)\; e^{\sim}$$

$$G(k,\tau) = \frac{1}{\beta^2}\sum_{\omega_1\omega_2} -\left\langle e^{-i\omega_1\tau}\,\varphi_{\omega_1}\,\varphi^*_{\omega_2}\right\rangle$$

$$= \frac{1}{\beta^2}\sum_{\omega_1\omega_2} e^{-i\omega_1\tau} \quad -\left\langle \varphi_{\substack{\omega_1\\k}}\,\varphi^*_{\substack{\omega_2\\k}}\right\rangle$$

$$\left\langle \varphi_{\substack{\omega_1\\k_0}}\,\varphi^*_{\substack{\omega_2\\k_0}}\right\rangle = \frac{1}{Z}\int \mathcal{D}\varphi^*\mathcal{D}\varphi \quad e^{\frac{1}{\beta}\sum_{\omega k}(i\omega-\xi_k)\varphi^*_{\omega k}\varphi_{\omega k}} \quad \varphi_{\substack{\omega_1\\k_0}}\,\varphi^*_{\substack{\omega_2\\k_0}}$$

$$= \frac{\beta}{i\omega_1-\xi_{k_0}}\,\delta_{\omega_1\omega_2}.$$

$$G(k_0,\tau) = \frac{1}{\beta}\sum_{\omega} e^{-i\omega\tau}\,\frac{1}{i\omega-\xi_{k_0}}.$$

$$\boxed{G^0(\omega_n,k_0) = \frac{1}{i\omega_n-\xi_{k_0}}}$$

Single particle Green's function.

$$G(k,\tau) = -\left\langle T_\tau\; b_k(\tau)\,b_k^+(0)\right\rangle$$

$$G(k,\tau=0^-) = -\left\langle b_k^+(0)\,b_k(\tau\to 0)\right\rangle \to -\left\langle b_k^+ b_k\right\rangle$$

$$G(k,\tau\to 0^-) \longrightarrow -\langle n_k\rangle$$

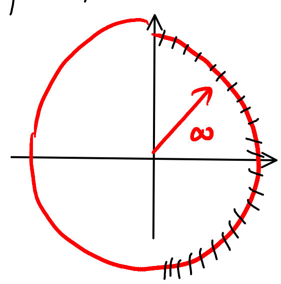
$$G(k, \tau \to 0^-) = \frac{1}{\beta} \sum_{\omega_n} e^{-i\omega_n \tau} \, G(k, \omega_n)$$

$$= \frac{1}{\beta} \sum_{\omega_n} e^{+i\omega_n 0^+} \, G(k, \omega_n)$$

$$= \frac{1}{\beta} \sum_{\omega_n} e^{i\omega_n 0^+} \frac{1}{i\omega_n - \xi_k} \qquad \omega_n = \frac{2\pi}{\beta} n.$$

$$\oint dz \, f(z) = 2i\pi \sum_{\text{residues}} f.$$

$$b(z) = \frac{1}{e^{\beta z} - 1} \qquad z_n = i \frac{2\pi}{\beta} n \qquad f(z_n) = \infty.$$



$$\oint dz \cdot e^{z 0^+} \frac{1}{z - \xi_k} \, b(z)$$

$$\frac{1}{e^{\beta z} - 1}.$$

$\operatorname{Re} z > 0$ $\quad b(z)$ exponentially small $\rightarrow \oint dz \ldots \rightarrow 0$

$\operatorname{Re} z < 0$ $\quad b(z) \rightarrow -1 \qquad e^{z 0^+} \rightarrow 0$ exponentially.

$$\oint dz \cdot e^{z\tau} \underbrace{\frac{1}{z - \xi_k}}_{\text{poles.}} \underbrace{b(z)}_{\text{poles.}} = 0 = 2i\pi \sum_{\text{residues}}$$

$$\frac{1}{e^{\beta z} - 1} \qquad \text{poles are } z = i \frac{2\pi}{\beta} n. \qquad z = i\frac{2\pi}{\beta} n + \delta z.$$

$$b(z_0 + \delta z) = \frac{1}{e^{\beta \delta z} - 1} \simeq \frac{1}{\beta \delta z}.$$

residue $\quad e^{i\omega_n \tau} \dfrac{1}{i\omega_n - \xi_k} \cdot \dfrac{1}{\beta}.$

$$\sum_{\text{residues}} = \frac{1}{\beta} \sum_{n} e^{i\omega_n \tau} \frac{1}{i\omega_n - \xi_k} = I.$$

$$\sum_{\substack{\text{residues}\\(\text{poles of } b)}} = \frac{1}{\beta}\sum_n e^{i\omega_n \tau}\frac{1}{i\omega_n - \xi_k} = I.$$

poles of $\frac{1}{z-\xi_k}$. residue: $e^{\xi_k \tau} \, b(\xi_k)$.

$$0 = I + e^{\xi_k \tau} \, b(\xi_k).$$

$$I = - b(\xi_k).$$

$$G(k, \tau = 0^-) \to - b(\xi_k)$$

3) Path integral for Fermions:

coherent states $c_\alpha |\Phi\rangle = \phi_\alpha |\Phi\rangle$

$c_\alpha c_\beta |\Phi\rangle = \phi_\alpha \phi_\beta |\Phi\rangle$ ϕ are numbers!

$c_\beta c_\alpha |\Phi\rangle = \phi_\beta \phi_\alpha |\Phi\rangle$

$c_\alpha c_\beta |\Phi\rangle = c_\beta c_\alpha |\Phi\rangle$

$\{c_\alpha, c_\beta\} = 0$ $c_\alpha c_\beta + c_\beta c_\alpha = 0$ $c_\alpha c_\beta = - c_\beta c_\alpha$.

Coherent state as defined for the bosons do not exist

[Negele + Orland. → details.]

$\phi_\alpha \phi_\beta = - \phi_\beta \phi_\alpha$. to build coherent states

Grassmann variables (Algebra).

$\phi_\alpha, \phi_\beta, \phi_\gamma$ .... $\phi_\alpha \phi_\beta = - \phi_\beta \phi_\alpha$. etc.

numbers.	Grassmann
$\phi_\alpha \phi_\beta = \phi_\beta \phi_\alpha$.	$\phi_\alpha \phi_\beta = - \phi_\beta \phi_\alpha$
$\int d\phi_\alpha$ is defined.	$\int d\phi_\alpha$ has to be defined.

$\int d\phi_\alpha$ is defined.

$\int d\phi_\alpha$ has to be defined.

$$f(\phi_\alpha, \phi_\beta, \phi_\gamma \dots) = C_1 + C_2\phi_\alpha + C_3\phi_\beta + \dots \phi_\alpha\phi_\beta$$

$$\cancel{\phi_\alpha^2}$$

$$\int d\phi_\alpha \, C_1 \qquad \int d\phi_\alpha \, \phi_\alpha .$$

$\to$ exactly identical to bosons **but**

$$\omega_n^{ferm.} = \frac{\pi}{\beta}(2p+1). \qquad \omega_n^{bos} = \frac{\pi}{\beta} 2p.$$

⚠ bosons

$$\begin{cases} \int du\,du^* \, e^{-\frac{1}{2} u_i^* M_{ij} u_j} = \frac{1}{\det M} \\ \dfrac{\int du\,du^* \, e^{-\frac{1}{2} u_i^* M_{ij} u_j} \, u_{i_0}^* u_{j_0}}{\int du\,du^* \, e^{-\frac{1}{2} u_i^* M_{ij} u_j}} = (M^{-1})_{i_0 j_0} \end{cases}$$

fermions

$$\begin{cases} \int dz\,dz^* \, e^{-\frac{1}{2} z_i^* M_{ij} z_j} = \det M \\ \dfrac{\int dz\,dz^* \, e^{-\frac{1}{2} z_i^* M_{ij} z_j} \, z_{i_0}^* z_{j_0}}{\int dz\,dz^* \, e^{-\frac{1}{2} z_i^* M_{ij} z_j}} = (M^{-1})_{i_0 j_0}, \end{cases}$$

Correlation functions.

A, B : fermion type, boson type. ?

C_α, C_β

⊖ fermion type

⊕ boson type. (ρ, j, $\sum_k \xi_k c_k^\dagger c_k$.

$$c_{r'} c_r = - c_r c_{r'}$$

$$\rho(r) = c_r^+ c_r \qquad \rho(r') = c_{r'}^+ c_{r'}$$

$$c_r^+ c_r \quad c_{r'}^+ c_{r'} \qquad\qquad c_{r'}^+ c_{r'} \; c_r c_r$$

$$\begin{aligned}
(c_r^+ c_r)(c_{r'}^+ c_{r'}) &= c_r^+ \left[\delta_{rr'} - c_{r'}^+ c_r \right] c_{r'} \\
&= c_r^+ \delta_{rr'} c_{r'} - c_r^+ c_{r'}^+ c_r c_{r'} \\
&= c_r^+ c_{r'} \delta_{rr'} + c_{r'}^+ c_r^+ c_r c_{r'} \\
&= c_r^+ c_{r'} \delta_{rr'} - c_{r'}^+ c_r^+ c_{r'} c_r . \\
&= c_r^+ c_{r'} \delta_{rr'} - c_{r'}^+ \left[\delta_{rr'} - c_{r'} c_r^+ \right] c_r \\
&= c_r^+ c_{r'} \delta_{rr'} - c_{r'}^+ c_r \delta_{rr'} \\
&\quad + (c_{r'}^+ c_{r'})(c_r^+ c_r)
\end{aligned}$$

$$\begin{aligned}
T_\tau \; A(\tau_1) B(\tau_2) &= A(\tau_1) \; B(\tau_2) \qquad \tau_1 > \tau_2 \\
&= \mp \; B(\tau_2) \; A(\tau_1) \qquad \tau_1 < \tau_2 .
\end{aligned}$$

$-$: for operators of fermion type.
$+$: for operators of boson type

$$G(k,\tau) = - \langle T_\tau \; c_k(\tau) \; c_k^+(0) \rangle$$

<u>bosons</u> $\quad G_b(k, \tau = 0^-) = - \langle b_k^+(0) \, b_k(\tau = 0^-) \rangle = - \langle b_k^+ b_k \rangle$

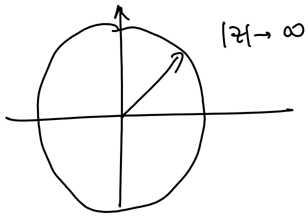
<u>fermions</u> $\quad G_f(k, \tau = 0^-) = + \langle c_k^+(0) \, c_k(\tau = 0^-) \rangle = + \langle c_k^+ c_k \rangle$

$$G_f(k, \tau = 0^-) = \frac{1}{\beta} \sum_{\omega_n} e^{i\omega_n 0^+} \frac{1}{i\omega_n - \xi_k.}$$

$$\omega_p = \frac{\pi}{\beta}(2p+1)$$

$$\frac{1}{\quad} . \qquad z = \frac{i\pi}{\beta}(2p+1) \qquad e^{i\pi(2p+1)} = e^{i\pi} = -1$$

$$\frac{\quad}{e^{\beta z}+1}$$



$$\oint dz \; e^{z0^+} \frac{1}{z-\xi_k} f(z) = 0$$

$$= 2\pi \left[\sum_{\text{poles } f} + \sum_{\substack{\text{other} \\ \text{poles}}} \right]$$

$$z = i\omega_p + \delta z . \qquad \frac{1}{e^{\beta(i\omega_p)} e^{\beta\delta z}+1} = \frac{1}{-e^{\beta\delta z}+1} = \frac{-1}{\beta\delta z}$$

$$\sum_{\text{pole } f} = -\frac{1}{\beta} \sum_{\omega_n = \frac{\pi}{\beta}(2p+1)} e^{i\omega_n 0^+} \frac{1}{i\omega_n - \xi_k} = -I .$$

$$0 = -I + f(\xi_k) \qquad I = f(\xi_k) .$$

Analytic continuation.

bosons $\quad G(\omega_n, k) = -\langle T_\tau \, A(\tau) \, B(0) \rangle$

$\downarrow \; i\omega_n \to \omega + i\delta .$

$$G_{ret}(\omega, k) = -i\,\Theta(t) \langle [A(\tau), B(0)] \rangle$$

fermions.

operators of bosonic type

$\hookrightarrow$ identical. $\quad [\; \omega_n = \frac{2\pi}{\beta} p .]$

analytic continuation $\quad G(\omega_n, k) \to G_{ret}(\omega_n, k)$

$$= -i\Theta(t) \langle [A(t), B(0)] \rangle$$

Operators of fermionic type

Matsubara frequencies: $\quad \omega_p = \frac{\pi}{\beta}(2p+1)$

$G(\omega_p, k) \to$

$$G(\omega_p, k) \;\to$$

$$-\langle T_\tau\, A^f(\tau) B^f(0) \rangle \;\to\; -i\,\theta(t)\,\langle \{ A^f(t), B^f(0) \} \rangle$$

Single particle Green's function.

$$G(\tau, k) = -\langle T_\tau\, c_k(\tau)\, c_k^\dagger(0) \rangle$$

$$G^0(\omega_p, k) = \frac{1}{i\omega_p - \xi_k} \qquad \text{(free fermions).}$$

$$G_{ret}(t, k) = -i\,\theta(t)\,\langle \{ c_k(t), c_k^\dagger(0) \} \rangle$$

IV.] Perturbation theory: Feynman diagrams.

$$H = \sum_k \xi_k\, c_k^\dagger c_k + \frac{1}{2} \sum_{\substack{k k' \\ q}} \tilde{V}(q)\, c_{k+q}^\dagger\, c_{k'-q}^\dagger\, c_{k'}\, c_k .$$

$$\chi(\tau, q) = -\langle T_\tau\, \rho(q,\tau)\, \rho^*(q, \tau=0) \rangle$$

$$\rho(q) = \sum_k c_{k+q}^\dagger c_k . \qquad \text{boson type}$$

$$\chi(\tau, q) = \frac{\int \mathcal{D}\phi^* \mathcal{D}\phi \;\; \rho(q,\tau)\, \rho^*(q,0)\;\; e^{S}}{\int \mathcal{D}\phi^* \mathcal{D}\phi \;\; e^{S}}$$

$$S = \int_0^\beta d\tau \sum_k \Big[\partial_\tau \phi_k^*(\tau)\, \phi_k(\tau) - H\big[\underset{c^\dagger}{\phi^*(\tau)}, \underset{c}{\phi(\tau)}\big] \Big]$$

$$= \underbrace{\frac{1}{\beta} \sum_k \sum_{\omega_n} (i\omega_n - \xi_k)\, \phi_{k\omega_n}^* \phi_{k\omega_n}}_{S_0} - \frac{1}{2} \sum_{\substack{k, k', q \\ \omega, \omega', \nu}} \phi_{k+q,\omega+\nu}^*\, \phi_{k'-q,\omega'-\nu}^*\, \phi_{k',\omega'}\, \phi_{k,\omega}$$

$$\frac{\int \mathcal{D}\phi^* \mathcal{D}\phi \quad \rho(\,)\rho(\,) \; e^{S_0} \left[1 + S_1 + \frac{1}{2!} S_1 S_1 + \frac{1}{3!} S_1 S_1 S_1 \right]}{\int \mathcal{D}\phi^* \mathcal{D}\phi \quad e^{S_0} \left[1 + S_1 + \frac{1}{2!} S_1 S_1 + \frac{1}{3!} S_1 S_1 S_1 + \dots \right]}$$

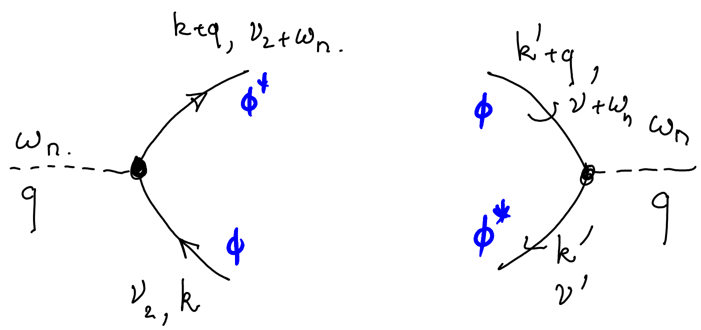
$$\langle \underbrace{\phi^*_{\alpha_1} \phi_{\alpha_2}}_{\frac{1}{i\omega_n - \xi_k}} \underbrace{\phi^*_{\alpha_3} \phi_{\alpha_4}}_{\frac{1}{i\omega_n - \xi_n}} \phi^*_{\alpha_5} \phi^*_{\alpha_6} \phi_{\alpha_7} \dots \rangle$$

$$\rho(\tau) = \sum_k \phi^*_{k+q}(\tau)\, \phi(\tau).$$

$$\rho(\omega_n) = \int_0^\beta d\tau \; e^{i\omega_n \tau} \sum_{k} \phi^*_{k+q}(\tau)\, \phi_k(\tau)$$

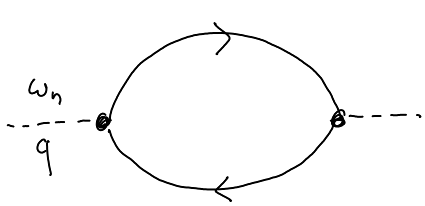
$$= \frac{1}{\beta^2} \sum_{\nu_1 \nu_2} \int_0^\beta d\tau \; e^{i\omega_n \tau} \; e^{-i\nu_1 \tau} \; e^{i\nu_2 \tau} \sum_k \phi^*_{k+q,\nu_1} \; \phi_{k,\nu_2}$$

$$\rho(\omega_n, q). = \frac{1}{\beta} \sum_{\nu_2} \sum_k \; \phi^*_{k+q,\, \nu_2 + \omega_n} \; \phi_{k,\nu_2}$$



$$\sum_{\nu_2, k}$$

$$\sum_{\nu', k'}$$

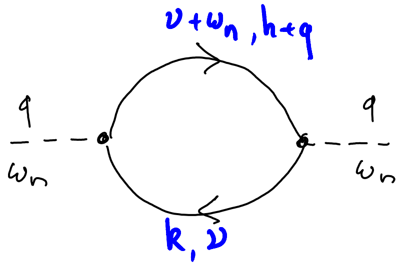


Write all the planar, topologically distinct diagrams -
Connecting the arrows

Connecting the arrows

- Replace lines by $\dfrac{1}{i\omega_n - \xi_k}$.

- Sum over all possible frequencies and momenta



$$\sum_{k,\nu} \frac{1}{i\omega_n + i\nu - \xi_{k+q}} \frac{1}{i\nu - \xi_k}$$

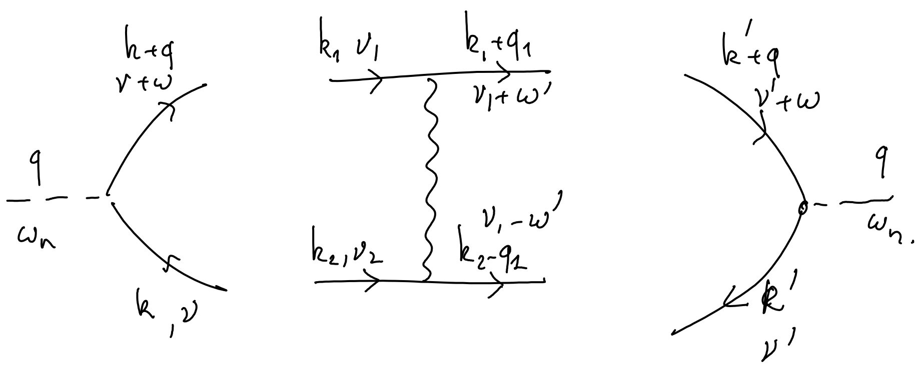
ω_n : even $\left(\frac{2\pi}{\beta} p\right)$ frequency

fermions : ν are odd frequencies.

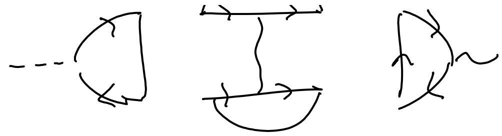
bosons : ν are even frequencies.

(Normalization : $\frac{1}{\Omega} \sum_k$ $\frac{1}{\beta} \sum_\nu$).

Order 1



disconnected diagrams



disconnected diagrams
(Cancelled by the denominator)

all the topologically distinct, <u>Connected</u> diagrams,

$$\overset{0}{\chi}(q,\omega_n) = \frac{1}{\beta\Omega}\sum_{k,\nu} \frac{1}{i\nu + i\omega_n - \xi_{k+q}} \frac{1}{i\nu - \xi_k}.$$

fermions. $\nu = \frac{\pi}{\beta}(2p+1)$

$$\oint dz \frac{1}{z+i\omega_n - \xi_{k+q}} \frac{1}{z - \xi_k} f(z) = 0$$

$$= \sum_{\substack{\text{poles}\\ f}} + \sum_{\substack{\text{other}\\ \text{poles}}}$$

$$\underbrace{-\frac{1}{\beta}\sum_{\nu} \frac{1}{i\nu + i\omega_n - \xi_{k+q}} \frac{1}{i\nu - \xi_k}}_{-I} + \sum_{\substack{\text{other}\\ \text{poles}}}.$$

- $z = \xi_k$. $\qquad \dfrac{f(\xi_k)}{\xi_k + i\omega_n - \xi_{k+q}}$
- $z = \xi_{k+q} - i\omega_n \qquad \dfrac{f(\xi_{k+q} - i\omega_n)}{\xi_{k+q} - i\omega_n - \xi_k}.$

$$I = \frac{f(\xi_k) - f(\xi_{k+q} - i\omega_n)}{i\omega_n + \xi_k - \xi_{k+q}}.$$

$$\chi^0(q,\omega_n) = \frac{1}{\Omega}\sum_k \frac{f(\xi_k) - f(\xi_{k+q} - i\omega_n)}{i\omega_n + \xi_k - \xi_{k+q}}.$$

$$\downarrow \chi_{ret}(q, i\omega_n \to \omega + i\delta)$$

$$\frac{1}{e^{\beta\left[\xi_{k+q} + i\frac{\pi}{\beta}2n\right]}+1} \qquad f(\xi_k - i\omega_n) = f(\xi_k)$$

Do first Sum over all frequencies then analytic continuation

$$\chi^0(q,\omega_n) = \frac{1}{\Omega}\sum_{k} \frac{f(\xi_k) - f(\xi_{k+q})}{i\omega_n + \xi_k - \xi_{k+q}}$$

$$\downarrow$$

$$\chi^0_{ret}(q,\omega) = \frac{1}{\Omega}\sum_{k} \frac{f(\xi_k) - f(\xi_{k+q})}{\omega + \xi_k - \xi_{k+q} + i\delta}.$$

V] Interacting bosons, Bose condensates, Mott insulators

- $\to$ Lattice vibrations (phonons)
- $\to$ Excitons.
- $\to$ Cooper pairs.
- $\to$ Spin excitations (Magnons)
- $\to$ He^4 (Superfluid)
- $\to$ Cold atoms: [kinetic energy vs interactions - for bosons / fermions]

- A.J. Leggett : Superfluids and Superconductors
  Review of Modern physics RMP 73 (2001)

- Cold atoms

Stringari and Pitaevskii (Oxford) BEC

I. Bloch, J. Dalibard, W. Zwerger RMP 885 (2008)

1.) Basis of BEC.

Free bosons $\sum_k \xi_k b_k^+ b_k$.

$$\langle n_k \rangle = \frac{1}{e^{\beta(\xi_k - \mu)} - 1}$$

$$N_{tot} = \sum_k n_k \to \int \frac{d^d k}{(2\pi)^d} \frac{1}{e^{\beta(\xi_k - \mu)} - 1}$$

$$= \int d\xi \, \mathcal{N}(\xi) \frac{1}{e^{\beta(\xi - \mu)} - 1}.$$

Free particles $\xi_k = \frac{k^2}{2m}$. $\mathcal{N}(\xi)_{d=3} = \frac{m^{3/2}}{\sqrt{2}\, \pi^2 \hbar^3} \xi^{1/2}$.

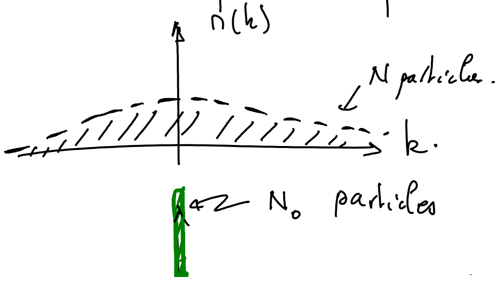
$$N_{tot}^{Max} = \int_0^{+\infty} d\xi \, \mathcal{N}(\xi) \frac{1}{e^{\beta \xi} - 1} = \int_0^{+\infty} d\xi \, \xi^{1/2} \frac{1}{e^{\beta\xi} - 1}$$

$$N \leqslant \int_0^{+\infty} d\xi \, \mathcal{N}(\xi) \frac{1}{e^{\beta\xi} - 1} = N_0 \quad \text{finite!}$$

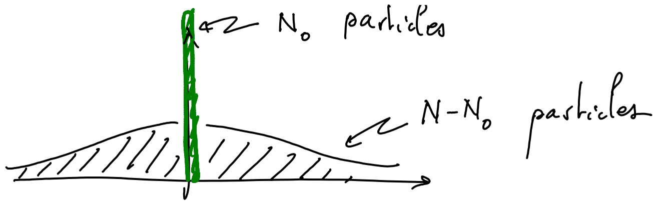
$N > N_0$

$$N = \sum_k \frac{1}{e^{\beta(\xi - \mu)} - 1}$$

Macroscopic occupation of $\underline{\underline{1}}$ quantum state.

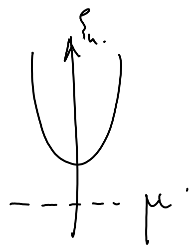


usual situation.



$T=0$, Free particles. All particles are in state $k=0$

$$N = \sum_k \frac{1}{e^{\beta(\xi_k-\mu)}-1} = \frac{1}{e^{-\beta\mu}-1} + \sum_{k\neq 0} \frac{1}{e^{\beta(\xi_k-\mu)}-1}$$



$$N = \frac{1}{e^{-\beta\mu}-1} + \int \frac{d^3k}{(2\pi)^3} \frac{1}{e^{\beta(\xi_k-\mu)}-1}$$

2.) Small interactions:

$$H = \sum_k (\xi_k-\mu) b_k^+ b_k + \frac{1}{2}\sum_{kk'q} \tilde{V}(q)\, b^+_{k+q} b^+_{k'-q} b_{k'} b_k .$$

$\langle n_k \rangle$ $\qquad N_{tot} = \sum_k \langle n_k\rangle$

1 Does BEC exist ?

2 How does interaction changes the condensate fraction

3 What are the excitations of the system

Bogoliubov approximation:

Macroscopic occupation of the state $k=0$.

$|\psi\rangle$ is the ground state.

$a_{k=0}|\psi\rangle \approx |\psi\rangle$ $\qquad \begin{cases} \langle\psi| a_{k=0} |\psi\rangle = \text{finite} \\ \langle\psi| a^+_{k=0} |\psi\rangle = \text{finite} \end{cases}$

Order parameter for the BEC

$\langle a_{k=0}\rangle \quad \langle a^+_{k=0}\rangle \neq 0 \qquad \langle a^+_{k=0}\rangle = \sqrt{n_0}.$

$$\langle a^{+}_{k=0} a_{k=0} \rangle \approx \langle a^{+}_{k=0} \rangle \langle a_{k=0} \rangle = n_0 .$$

$$n(k) = \langle a^{+}_{k} a_k \rangle .$$

$$\langle \psi^{+}(x)\, \psi(0) \rangle \qquad\qquad \langle \psi(x)\, \psi^{+}(0) \rangle$$

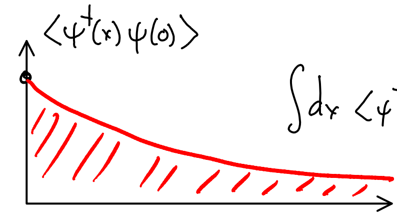
$$\int dx\; e^{ikx} \langle \psi^{+}(x)\, \psi(0) \rangle = \int dx \sum_{k_1 k_2} e^{ikx} \langle b^{+}_{k_1} b_{k_2} \rangle e^{-ik_1 x}$$

$$= \sum_{k_2} \langle b^{+}_{k} b_{k_2} \rangle = \langle b^{+}_{k} b_k \rangle$$

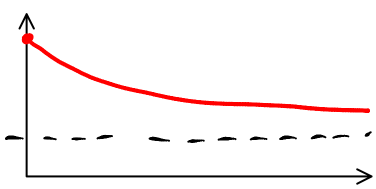
$n(k) \Longleftrightarrow$ Fourier transform of $\langle \psi^{+}(x)\, \psi(0) \rangle$

$$n(k) = \int dx\; e^{ikx} \langle \psi^{+}(x)\, \psi(0) \rangle$$

$$n(k=0) = \int dx\; \langle \psi^{+}(x)\, \psi(0) \rangle$$

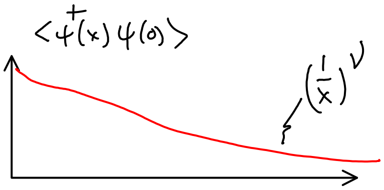


$$\int dx\, \langle \psi^{+}(x)\psi(0) \rangle = \text{finite.}$$ exponential decay



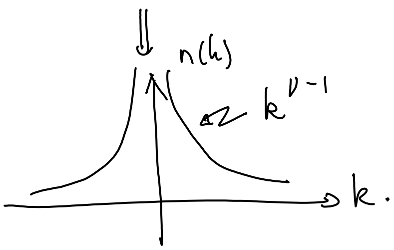
$$\langle \psi^{+}(x)\, \psi(0) \rangle \to n_0 .$$

$$\int dx\; e^{ikx}\, n_0 \to n_0\, \delta(k)$$



Slow (power law) decay

$$\int dx\; e^{ikx} \left(\frac{1}{x}\right)^{\nu}$$



$$\sim x^{1-\nu} \to \left(\frac{1}{k}\right)^{1-\nu} = k^{\nu-1}$$

$$\langle \psi^{+}(x)\, \psi(0) \rangle \xrightarrow[x\to\infty]{} \langle \psi^{+}(x) \rangle \langle \psi(0) \rangle = n_0 .$$

$$\langle \varphi^\dagger(x)\rangle \equiv \sqrt{n_0} = \langle \varphi(x)\rangle$$

$$\frac{1}{\Omega}\sum_{\substack{k k' \\ q}} b^\dagger_{k+q}\, b^\dagger_{k'-q}\, b_{k'}\, b_k\, \tilde V(q) \qquad \tilde V(q) \equiv V_0$$

$k=0$ state $\quad b^\dagger_k \to \langle b^\dagger_k\rangle \equiv \sqrt{n_0}$ ← very large

$$\frac{1}{\Omega}\Big[V_0\, n_0^2 + V_0\, n_0 \sum_q \left[b^\dagger_q\, b^\dagger_{-q} + b_{-q}\, b_q\right]$$

$k+q=0 \quad k=-q$

$k'-q=0 \quad k'=q$

$$+ V_0 n_0 \sum b^\dagger_k\, b_k \Big]$$

$$H \equiv \sum_k \left[(\xi_k - \mu) + V_0 \frac{n_0}{\Omega}\right] b^\dagger_k b_k + \frac{V_0 n_0}{\Omega}\sum_k \left(b^\dagger_k b^\dagger_{-k} + b_{-k} b_k\right)$$

$$+ \cdots$$

$$\begin{cases} \alpha_k = u_k\, b_k + v_k\, b^\dagger_{-k} \\ \beta_k = u_k\, b_{-k} + v_k\, b^\dagger_k \end{cases}$$

(Bogoliubov transformation)

- Depletion of the condensate by the interactions
- $d=3$ finite

  $d \leq 2$ total. (but quasi-long range order)

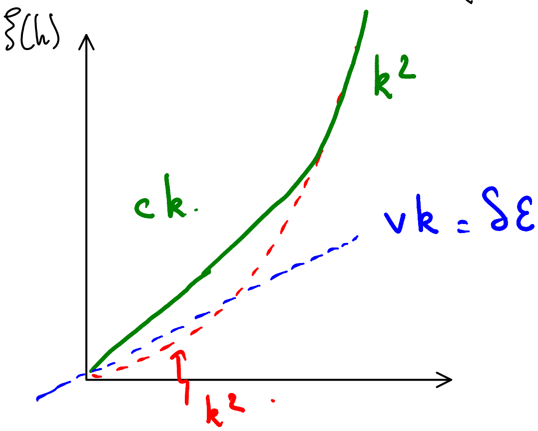
- <u>Spectrum of excitations.</u>

$$\xi(k) = \sqrt{(c\,k)^2 + \left(\frac{k^2}{2m}\right)^2} \qquad \begin{cases} c = \sqrt{\dfrac{V_0 \bar n_0}{m}} \\ \bar n_0 = \dfrac{n_0}{\Omega} \end{cases}$$

broken phase symmetry

$$\xi(k) = \sqrt{(ck)^2 + \left(\frac{k^2}{2m}\right)^2}$$

$$\begin{cases} c = \sqrt{\dfrac{V_0 \bar{n}_0}{m}} \\ \bar{n}_0 = \dfrac{n_0}{\Omega} \end{cases}$$

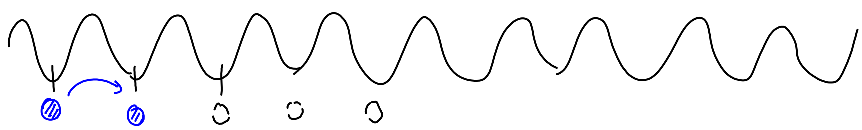


broken phase symmetry

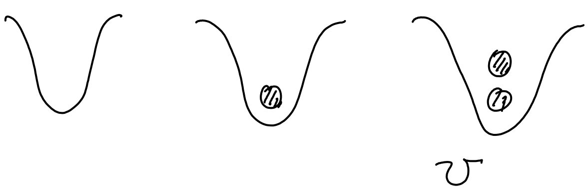
v smaller than c : no excitations possible $\Rightarrow$ Superfluid.

$v > c$ $\rightarrow$ above the critical current : dissipation.

3) Strong interactions. bosons of lattice.



$$H = -t \sum_{\langle ij \rangle} \left(b_i^{\dagger} b_j + b_j^{\dagger} b_i \right) + \frac{U}{2} \sum_i n_i (n_i - 1)$$



Bose-Hubbard model.

$t \gg U$ $\Rightarrow$ Superfluidity

$t \ll U$ $\Rightarrow$ interaction effect dominate

$n = 1$ $\Rightarrow$ Mott isolator of bosons.