APPENDIX E

SINE-GORDON

E.1 Renormalization

Let me give an alternative way to renormalize the sine-Gordon Hamiltonian. This method is more Wilson-like than the one using the correlation functions. We start from the action deriving from (4.17) (see also (3.26) for the quadratic part)

$$S = \frac{1}{2\pi K}\int dx\, d\tau\, [\frac{1}{u}(\partial_\tau\phi)^2 + u(\partial_x\phi)^2] + \frac{2g}{(2\pi\alpha)^2}\int dx\, d\tau\, \cos(\sqrt{8}\phi(x,\tau)) \quad \text{(E.1)}$$

The field $\phi(x,\tau)$ can be written in terms of the Fourier modes

$$\phi(x,\tau) = \frac{1}{\beta\Omega}\sum_{k,\omega_n} e^{i(kx-\omega_n\tau)}\phi(k,\omega_n) \quad \text{(E.2)}$$

For simplicity, I assume here $\beta = \infty$. Let me impose a sharp momentum cutoff Λ to start with. If one varies the cutoff between Λ and Λ' one can decompose ϕ in fast and slow Fourier modes ($r = (x, u\tau)$ and $\mathbf{q} = (k, \omega_n/u)$):

$$\phi(r) = \phi^>(r) + \phi^<(r) \quad \text{(E.3)}$$

where

$$\begin{aligned}\phi^>(r) &= \frac{1}{\beta\Omega}\sum_{\Lambda'<||\mathbf{q}||<\Lambda} e^{i\mathbf{q}\cdot r}\phi(q) \\ \phi^<(r) &= \frac{1}{\beta\Omega}\sum_{||\mathbf{q}||<\Lambda'} e^{i\mathbf{q}\cdot r}\phi(q)\end{aligned} \quad \text{(E.4)}$$

with the notation (A.5). The quadratic part of the action

$$S_0 = \frac{1}{2\pi K}\frac{1}{\beta\Omega}\sum_{\mathbf{q}}[\omega_n^2/u + uk^2]\phi(\mathbf{q})^*\phi(\mathbf{q}) \quad \text{(E.5)}$$

can obviously be written as

$$S_0 = S_0^> + S_0^< \quad \text{(E.6)}$$

The partition function can be expanded in powers of the cosine term, which gives up to second order (Z_0 is the partition function for $g = 0$)

$$\frac{Z}{Z_0} = \frac{1}{Z_0}\int \mathcal{D}\phi e^{-S_0^> - S_0^<}\Bigg[1 - \frac{2g}{(2\pi\alpha)^2 u}\int d^2r\ \cos(\sqrt{8}(\phi^>(r) + \phi^<(r)))$$
$$+\frac{2g^2}{(2\pi\alpha)^4u^2}\int d^2r_1 \int d^2r_2\ \cos(\sqrt{8}(\phi^>(r_1)+\phi^<(r_1)))\cos(\sqrt{8}(\phi^>(r_2)+\phi^<(r_2)))\Bigg] \tag{E.7}$$

One can make the average over the fast modes in order to get an effective action for the slow modes. This gives

$$\frac{Z}{Z_0} = \frac{1}{Z_0^<}\int \mathcal{D}\phi\, e^{-S^<}\Bigg[1 - \frac{2g}{(2\pi\alpha)^2 u}\int d^2r\ \cos(\sqrt{8}\phi^<(r))e^{-4\langle(\phi^>(r))^2\rangle_>}]$$
$$+\frac{g^2}{(2\pi\alpha)^4u^2}\sum_{\epsilon=\pm}\int d^2r_1 \int d^2r_2\ \cos(\sqrt{8}(\phi^<(r_1)+\epsilon\phi^<(r_2)))e^{-4\langle(\phi^>(r_1)+\epsilon\phi^>(r_2))^2\rangle_>}\Bigg] \tag{E.8}$$

In order to get an effective action one can reexponentiate this expression. This is equivalent to the standard cumulant expansion. One obtains

$$\frac{Z}{Z_0} = \frac{1}{Z_0^<}\int \mathcal{D}\phi\, e^{-S^< - \frac{2g}{(2\pi\alpha)^2u}\int d^2r \cos(\sqrt{8}\phi^<(r))e^{-4\langle\phi^>(r)\rangle_>}}$$
$$e^{\frac{g^2}{(2\pi\alpha)^4u^2}\int d^2r_1\int d^2r_2[\sum_{\epsilon=\pm}\cos(\sqrt{8}(\phi^<(r_1)+\epsilon\phi^<(r_2)))e^{-4\langle(\phi^>(r_1)+\epsilon\phi^>(r_2))^2\rangle_>}]}$$
$$e^{-\frac{2g^2}{(2\pi\alpha)^4u^2}\int d^2r_1\int d^2r_2\cos(\sqrt{8}\phi^<(r_1))e^{-4\langle\phi^>(r_1)\rangle_>}\cos(\sqrt{8}\phi^<(r_2))e^{-4\langle\phi^>(r_2)\rangle_>}} \tag{E.9}$$

The first term is like the original cosine but for the slow fields only, that is, with a smaller cutoff Λ'. To get back to an action identical to the original one has to rescale distance and time to bring back the cutoff to its original value. This can be done by defining

$$dk = \frac{\Lambda'}{\Lambda}dk' \tag{E.10}$$

and the same transformation for ω. This ensures that the new variables have the old cutoff Λ. In real space this means that one should rescale space and time according to

$$dx = \frac{\Lambda}{\Lambda'}dx', \quad d\tau = \frac{\Lambda}{\Lambda'}d\tau' \tag{E.11}$$

After this rescaling one recovers a theory that is exactly identical to the original one but with a new coupling constant

$$g(\Lambda') = \left(\frac{\Lambda}{\Lambda'}\right)^2 g(\Lambda)e^{-4\langle(\phi^>(r))^2\rangle_>} = \left(\frac{\Lambda}{\Lambda'}\right)^2 g(\Lambda)e^{-\frac{4}{\beta\Omega}\sum_{\Lambda'<||\mathbf{q}||<\Lambda}\frac{\pi K u}{\omega_n^2+u^2k^2}} \tag{E.12}$$

For $\beta \to \infty$ and $L \to \infty$ the sum can be converted to a two-dimensional integral

$$\begin{aligned} g(\Lambda') &= \left(\frac{\Lambda}{\Lambda'}\right)^2 g(\Lambda) e^{-2\int_{\Lambda'<||q||<\Lambda} dq \frac{K}{q}} \\ &= \left(\frac{\Lambda}{\Lambda'}\right)^2 g(\Lambda) e^{-2K\int_{\Lambda'}^{\Lambda} \frac{dq}{q}} = \left(\frac{\Lambda}{\Lambda'}\right)^2 g(\Lambda) e^{-2K\log(\Lambda/\Lambda')} \end{aligned} \tag{E.13}$$

If one parametrizes the cutoff as usual by $\Lambda(l) = \Lambda_0 e^{-l}$ where Λ_0 is the bare cutoff, and one makes the infinitesimal change $\Lambda' = \Lambda_0 e^{-l-dl}$, one gets

$$g(l + dl) = g(l) e^{(2-2K)dl} \tag{E.14}$$

which gives the renormalization equation

$$\frac{dg(l)}{dl} = g(l)(2 - 2K(l)) \tag{E.15}$$

This is identical to (2.134) obtained by a direct renormalization of the correlation functions.

In the term of order g^2 in (E.9) the last term is there to cancel the disconnected parts (that is, the parts for which the points r_1 and r_2 are very far from each other). The main contribution thus comes from the region where the two points r_1 and r_2 are close. There are thus two main contributions depending on the sign of ϵ. If $\epsilon = +1$ the term $\cos(\sqrt{8}(\phi_1 + \phi_2))$ is essentially a term $\cos(2\sqrt{8}\phi(r))$ since we want the two points to be close $r_1 \sim r_2 \sim r$. This term is a new cosine term with an argument that is twice that of the original cosine. We can thus expect this new term to be less relevant than the original cosine since its coupling constant will renormalize with an equation similar to (E.15) but with $8K$ instead of $2K$. Close to the point where the original cosine becomes relevant, $K \sim 1$, one can throw away this contribution. We can thus only retain the contribution with $\epsilon = -1$. One can rewrite this part as

$$\begin{aligned} \delta I = \frac{g^2}{(2\pi\alpha)^4 u^2} \int d^2 r_1 \int d^2 r_2 \cos(\sqrt{8}(\phi^<(r_1) - \phi^<(r_2))) \\ [e^{-4\langle(\phi^>(r_1)-\phi^>(r_2))^2\rangle_>} - e^{-8\langle(\phi^>)^2\rangle_>}] \end{aligned} \tag{E.16}$$

This can be rewritten

$$\begin{aligned} \delta I =& \frac{g^2}{(2\pi\alpha)^4 u^2} \int d^2 r_1 \int d^2 r_2 \cos(\sqrt{8}(\phi^<(r_1) - \phi^<(r_2))) \\ & \left[e^{-\frac{4}{\beta\Omega}\sum_{\Lambda'<||\mathbf{q}||<\Lambda}[2-2\cos(\mathbf{q}r)]\frac{\pi K u}{\omega_n^2+u^2k^2}} - e^{-\frac{4}{\beta\Omega}\sum_{\Lambda'<||\mathbf{q}||<\Lambda}[2]\frac{\pi K u}{\omega_n^2+u^2k^2}} \right] \\ =& \frac{g^2}{(2\pi\alpha)^4 u^2} \int d^2 r_1 \int d^2 r_2 \cos(\sqrt{8}(\phi^<(r_1) - \phi^<(r_2))) \\ & e^{-\frac{4}{\beta\Omega}\sum_{\Lambda'<||\mathbf{q}||<\Lambda}[2-2\cos(\mathbf{q}r)]\frac{\pi K u}{\omega_n^2+u^2k^2}} \left(1 - e^{-\frac{4}{\beta\Omega}\sum_{\Lambda'<||\mathbf{q}||<\Lambda}[2\cos(\mathbf{q}r)]\frac{\pi K u}{\omega_n^2+u^2k^2}}\right) \end{aligned} \tag{E.17}$$

where $r = r_1 - r_2$. Since the integral over $\mathbf{q}$ is only for values of the order of the cutoff this constrains r to be of the order of $1/\Lambda$. One can thus: (i) make an expansion of the cosine in powers of r were we introduce the center of mass $R = (r_1 + r_2)/2$ and relative coordinates $r = r_1 - r_2$; (ii) expand the exponential in the last term. Because the last term in (E.17) is proportional to dl all rescaling terms and terms that depend on Λ/Λ' can be replaced by 1. There is one problem however: the fluctuations of the field ϕ are unbounded ($\langle\phi^2\rangle = \infty$) thus one cannot expand the cosine directly (Nozieres and Gallet, 1987). To do it safely one needs to normal order the cosine. This can be done using

$$\cos(\phi) =: \cos(\phi) : e^{-\frac{1}{2}\langle\phi^2\rangle} \tag{E.18}$$

The normal ordered cosine can be expanded safely. Thus,

$$\cos(\sqrt{8}(\phi^<(r_1) - \phi^<(r_2))) \simeq 4(r \cdot \nabla_R \phi(R))^2 e^{-\frac{4}{\beta\Omega}\sum_{||\mathbf{q}||<\Lambda'}[2-2\cos(\mathbf{q}r)]\frac{\pi K u}{\omega_n^2+u^2k^2}} \tag{E.19}$$

The exponential term in (E.19) exactly combines with the corresponding exponential term in (E.17). Using

$$\frac{4}{\beta\Omega}\sum_{\Lambda'<||\mathbf{q}||<\Lambda}[2\cos(\mathbf{q}r)]\frac{\pi K u}{\omega_n^2 + u^2k^2} = 4K\int_{\Lambda'}^{\Lambda}\frac{dq}{q}J_0(qr) \tag{E.20}$$

one thus has

$$\begin{aligned}\delta I =& \frac{g^2 16 K dl}{(2\pi\alpha)^4 u^2}\int d^2R\int d^2r (r\cdot\nabla_R\phi(R))^2 e^{-4KF_{1,\Lambda}(r)}J_0(\Lambda r)\\ =& dl\frac{g^2 8K}{(2\pi\alpha)^4u^2}\int d^2R\left[(\partial_X\phi)^2 + (\partial_Y\phi)^2\right]\left[\int d^2r r^2 e^{-4KF_{1,\Lambda}(r)}J_0(\Lambda r)\right]\end{aligned} \tag{E.21}$$

where we have expanded the last term in (E.17) and

$$F_{1,\Lambda}(r) = \frac{1}{\beta\Omega}\sum_{||\mathbf{q}||<\Lambda}[2 - 2\cos(\mathbf{q}r)]\frac{\pi u}{\omega_n^2 + u^2k^2} = \int_0^{\Lambda}\frac{dq}{q}[1 - J_0(qr)] \tag{E.22}$$

The term δI is thus a correction to the coefficient $1/(2\pi K)$ in the quadratic action. The velocity u is not renormalized, which is a consequence of the Lorentz invariance of the action. This leads to the renormalization equation

$$\frac{dK^{-1}(l)}{dl} = \frac{g^2 8K(l)}{(2\pi)^2(\Lambda\alpha)^4u^2}\Lambda^4\int_0^{\infty}dr\, r^3 e^{-4KF_{1,\Lambda}(r)}J_0(\Lambda r) \tag{E.23}$$

one can simply rescale the integral in (E.23) to make the Λ dependence disappear. The RG equation thus becomes

$$\frac{dK^{-1}(l)}{dl} = \frac{g^2 2K(l)}{(\pi u)^2(\Lambda\alpha)^4}\mathrm{C} \tag{E.24}$$

where

$$C = \int_0^\infty dz\, z^3 e^{-4KF_1(z)} J_0(z) \tag{E.25}$$

Equation (E.24) is essentially identical to (2.134). The difference between (E.24) and (2.134) in the precise determination of the dimensionless coupling constant for the renormalization equation of K comes from the fact that different cutoff procedures have been used in the two cases. Here we have used a hard cutoff in momentum space, whereas a hard cutoff in real space was imposed in (2.134). Clearly, C and hence the RG equations depend on the precise cutoff procedure used. Of course, the physical quantities are independent of this cutoff procedure. With a generalization of the procedure exposed in this appendix it is possible to derive the RG equations for an arbitrary cutoff (Nozieres and Gallet, 1987).

E.2 Variational calculation

Let us now examine another method, which can be useful even in the absence of a small parameter in the Hamiltonian. It is less powerful than the RG to give the critical properties of the system,[41] but is extremely useful to have the physics of the massive phases, for which the RG would stupidly flow to strong coupling. This is the standard variational method (Feynman, 1972).

Quite generally one can write

$$Z = \int \mathcal{D}\phi e^{-S} = \int \mathcal{D}\phi e^{-S_0} e^{-(S-S_0)} = Z_0 \langle e^{-(S-S_0)} \rangle_0 \tag{E.26}$$

where the index 0 denoted the partition function and the averages with respect of an action S_0. Here, S_0 can be any action. Thus, the free energy satisfies

$$F = F_0 - T\log[\langle e^{-(S-S_0)}\rangle_0] \tag{E.27}$$

Given the convexity of the exponential (see, e.g. Feynman 1972) it is easy to check that one has always

$$\langle e^{-(S-S_0)} \rangle > e^{-\langle (S-S_0) \rangle} \tag{E.28}$$

and thus

$$F \leq F_{\mathrm{var}} = F_0 + T\langle S - S_0 \rangle_0 \tag{E.29}$$

The 'best' S_0 is obviously S, that is, the one for which the variational free energy F_{var} is the exact one, that is, minimum. The idea is to take a simple enough S_0 so that one can compute (e.g. a quadratic one), and to optimize it to try to get as close as possible of the physics of the original system.

[41]Nothing can beat the RG for that.

Let us take an example. We start from our favorite sine-Gordon action (see Section 2.3.2).

$$S = \frac{1}{2\pi K}\int dx\, d\tau\, [\frac{1}{u}(\partial_\tau\phi)^2 + u(\partial_x\phi)^2] - \frac{2g}{(2\pi\alpha)^2}\int dx\, d\tau\, \cos(\sqrt{8}\phi) \quad (E.30)$$

Given the cosine term, the optimal classical configuration (that is, the one giving the minimum of the action) would correspond to $\phi = 0$. One can thus reasonably expect that an approximation where one takes into account harmonic oscillations around this equilibrium position is a good one. Let us thus take for S_0

$$S_0 = \frac{1}{2\beta\Omega}\sum_{\mathbf{q}} G^{-1}(\mathbf{q})\phi^*(\mathbf{q})\phi(\mathbf{q}) \quad (E.31)$$

and try to optimize by finding the best Green's function $G(q)$. Here, we have *a priori* an infinite number of variational parameters (one for each q). The variational energy is

$$F_{\text{var}} = -T\sum_{q>0}\log[G(q)] + \frac{T}{2\pi K}\sum_q[\frac{1}{u}\omega_n^2 + uk^2]G(q) - T\frac{2g}{(2\pi\alpha)^2}\beta\Omega e^{-\frac{4}{\beta\Omega}\sum_{\mathbf{q}} G(\mathbf{q})} \quad (E.32)$$

I did not write the term $\langle S_0\rangle_{S_0}$, which gives a simple constant and does not contribute to the variational equations. The optimal $G(q)$ obeys

$$\frac{\partial F_{\text{var}}}{\partial G(q)} = 0 \quad (E.33)$$

This leads to

$$G^{-1}(q) = \frac{1}{\pi K}[\frac{1}{u}\omega_n^2 + uk^2] + \frac{16g}{(2\pi\alpha)^2}e^{-\frac{4}{\beta\Omega}\sum_{\mathbf{q}} G(\mathbf{q})} \quad (E.34)$$

which is a self-consistent equation for $G(q)$. Obviously, one can write

$$G^{-1}(\mathbf{q}) = \frac{1}{\pi K}[\frac{1}{u}\omega_n^2 + uk^2 + \frac{\Delta^2}{u}] \quad (E.35)$$

where the gap Δ satisfies

$$\frac{\Delta^2}{\pi K u} = \frac{16g}{(2\pi\alpha)^2}e^{-\frac{4}{\beta\Omega}\sum_{\mathbf{q}}\frac{\pi K u}{\omega_n^2 + u^2k^2 + \Delta^2}} \quad (E.36)$$

Let us look at the zero temperature limit $\beta \to \infty$ and the thermodynamic limit. We transform as usual the sum into a two-dimensional integral. To avoid an unphysical ultraviolet divergence one has again to put a large momentum cutoff

Λ. In the absence of the gap Δ the integral would be divergent at small q (infrared divergence).

$$\frac{4}{(2\pi)^2}\int d\mathbf{q}\frac{\pi K u}{\omega^2+u^2k^2+\Delta^2} = 2K\int_0^\Lambda q\,dq\,\frac{1}{q^2+(\Delta/u)^2} \simeq 2K\log[u\Lambda/\Delta] \tag{E.37}$$

assuming that $\Delta \ll u\Lambda$. One has thus the selfconsistent equation

$$\Delta^2 = \frac{4Ku^2y}{\alpha^2}\left(\frac{\Delta}{u\Lambda}\right)^{2K} \tag{E.38}$$

It is easy to see that for $K > 1$ this equation has only $\Delta = 0$ for solution. The sine-Gordon system behaves as a free theory and the cosine potential is irrelevant. If, on the other hand, $K < 1$ a non-zero solution appears for Δ

$$\Delta = u\Lambda\left(\frac{4Ky}{(\alpha\Lambda)^2}\right)^{\frac{1}{2-2K}} \tag{E.39}$$

One essentially recovers the value of the gap (2.158) that we had obtained deep in the massive phase. The physics of the massive phase as given by the variational approach is essentially the one we had discussed from physical arguments. The field ϕ is trapped in one of the minima and makes small oscillations around this minima. Such a method is thus very useful to compute physical properties in the massive phases.

Two warnings in using this variational method. First, it obviously missed the correct critical properties. The way the gap goes to zero at the transition is not correct. It is as if the variational method was replacing the correct RG flow by a purely vertical flow. This is not a very serious drawback since any approximate method is not expected to capture critical behaviors. A more serious caveat is that the variational approach can only capture the small oscillations around the minima. It misses other excitations such as solitons where the field ϕ goes from one of the minima to the other. These excitations can be very important so caution should be exerted to check that the variational approach did not miss the essential physics for the problem at hand. Despite these two caveats this is a very useful approach.

E.3 Semiclassical approximations

I just briefly recall some results obtained for the sine-Gordon model. A complete description of these results is given in Rajaraman (1982). I take the model

$$H = \frac{1}{2\pi}\int dx\; uK(\pi\Pi(x))^2 + \frac{u}{K}(\nabla\phi(x))^2 + \overline{g}\int dx[1-\cos(\sqrt{8}\phi(x))] \tag{E.40}$$

so that the minimum of energy corresponds to $\phi = 0$.