HOW TO BUILD A QUANTUM COMPUTER (in principle!)

[Note: Ensuing discussion refers to a "circuit-based" QC which could in principle implement (e.g.) Shor's factoring algorithm. Requirements for other kinds of application, e.g. quantum annealing, are not necessarily quite so stringent in all respects].

First requirement: a set of N (typically »1) 2-state systems ("qubits") on which we can implement 1-qubit operations on (all) individual qubits and 2-qubit operations on (at least some) pairs of qubits.

(Why not 3, 4. . . –state systems?)

The "language" of qubits: the Bloch sphere

Consider an arbitrary quantum system restricted to move in a 2D Hilbert space. Let's choose a definite basis in this space and label the axes (arbitrarily) $|0\rangle$ and $|1\rangle$. Then an arbitrary pure state of the system can be written

$$|\Psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \qquad \text{with } \underbrace{|\alpha|^2 + |\beta|^2 = 1}_{\text{normalization}}$$

Moreover, an arbitrary Hermitian operator $\hat{\Omega}$ in the space may be written in matrix form as

$$\hat{\Omega} \equiv \begin{pmatrix} \Omega_{00} & \Omega_{10} \\ \Omega_{10} & \Omega_{11} \end{pmatrix}$$

Let us introduce the unit matrix $\hat{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the three standard Pauli matrices $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$ satisfying

$$Tr\hat{\sigma}_i = 0, \qquad \hat{\sigma}_i^2 = \hat{1}, \qquad \left\{\hat{\sigma}_i, \hat{\sigma}_j\right\} = 2\delta_{ij}, \qquad \left[\hat{\sigma}_i, \hat{\sigma}_j\right] = i\varepsilon_{ijk}\hat{\sigma}_k$$

Explicit form of the Pauli matrices (in the "standard" representation):

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now, we have

$$\hat{\Omega} \equiv \begin{pmatrix} \Omega_{00} & \Omega_{10}^* \\ \Omega_{10} & \Omega_{11} \end{pmatrix}$$

so if we define

$$\Omega \equiv \frac{1}{2}(\Omega_{00} + \Omega_{11}), \ \Omega_x \equiv \frac{1}{2}(\Omega_{10} + \Omega_{10}^*), \ \Omega_y \equiv \frac{1}{2}(\Omega_{10} - \Omega_{10}^*),$$

$$\Omega_z \equiv \frac{1}{2}(\Omega_{00} - \Omega_{11})$$

(i.e. $\Omega \equiv \frac{1}{2} Tr \hat{\Omega}\hat{1}, \ \Omega_i \equiv \frac{1}{2} tr \hat{\Omega}\hat{\sigma}_i$)

then any Hermitian operator $\hat{\Omega}$ can be represented in the form

$$\hat{\Omega} = \Omega\hat{1} + \underset{\sim}{\Omega} \cdot \underset{\sim}{\hat{\sigma}}$$

where Ω is a real scalar and $\underset{\sim}{\Omega}$ a real vector. Consequently, any qubit is isomorphic to a spin ½.

Since the density matrix $\hat{\rho}$ is a Hermitian operator with unit trace, it is a special case of (*) and can be written in the form $\hat{\rho} = \frac{1}{2}(1 + \varepsilon \underset{\sim}{n} \cdot \underset{\sim}{\hat{\sigma}})$

$\underset{\sim}{n} =$ unit vector, $\varepsilon \equiv 2|\underset{\sim}{\rho}|$)

Note that $Tr \ \hat{\rho}^2 = \frac{1}{2}(1 + \varepsilon^2)$

Since in general $Tr \ \hat{\rho}^2 \leq Tr \ \hat{\rho}$ with the equality holding (only) for a pure state, we conclude

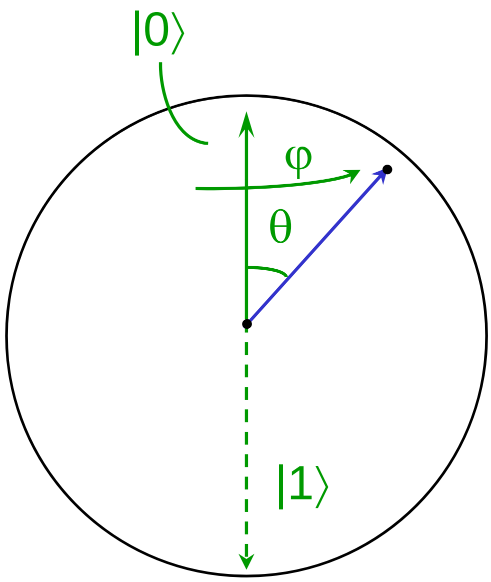
$\varepsilon = 1$ for a pure state,

$0 \leq \varepsilon < 1$ for a mixed state.

Thus,

Any pure state of a qubit is uniquely described by specifying the corresponding vector $\underset{\sim}{n}$ on the unit sphere ("Bloch sphere"). The physical significance of $\underset{\sim}{n}$ is that it is the direction along which the "spin" of the qubit "points," i.e.

$$\underset{\sim}{n}\cdot\hat{\underset{\sim}{\sigma}}=+1$$



Relation to the representation
$|\Psi\rangle=\alpha|0\rangle+\beta|1\rangle$:
$|0\rangle$ obviously satisfies

$$\sigma_z|0\rangle=+|0\rangle$$

so corresponds to $\underset{\sim}{n}=\underset{\sim}{z}$ (N. pole)

Similarly, $|1\rangle$ satisfies $\sigma_z|1\rangle=-|1\rangle$
so corresponds to $\underset{\sim}{n}=-\underset{\sim}{z}$ (S. pole)
What about the general case? Writing

$$\underset{\sim}{n}\cdot\hat{\underset{\sim}{\sigma}}|\Psi\rangle=|\Psi\rangle, \qquad |\Psi\rangle=\alpha|0\rangle+\beta|1\rangle$$

we find explicitly

$$\alpha/\beta=\frac{\sin\theta e^{-i\varphi}}{1-\cos\theta}\equiv\cot\frac{\theta}{2}e^{i\varphi}$$

so, up to an overall phase (physically irrelevant)

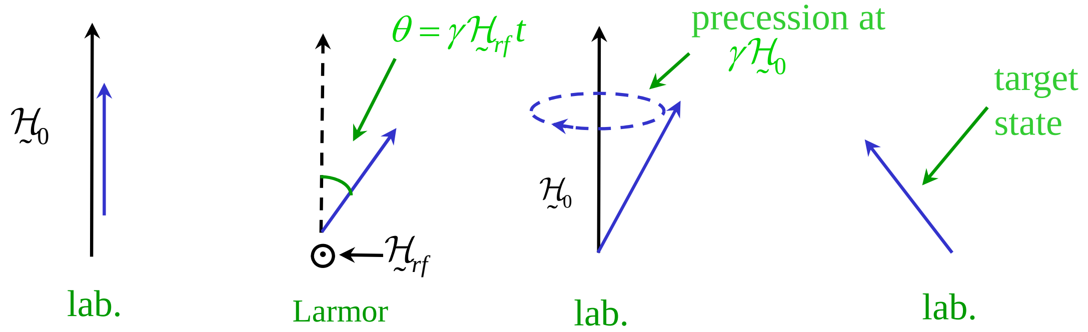
$$|\Psi\rangle=\cos\frac{\theta}{2}|0\rangle+\sin\frac{\theta}{2}e^{i\varphi}|1\rangle$$

Mixed state: $\varepsilon<1$, but (except for $\varepsilon=0$) still specified by unit vector $\underset{\sim}{n}$ on Bloch sphere. Can be represented as mixture of states polarized ± along $\underset{\sim}{n}$, with "polarization" $\left(\frac{1}{2}(P_+-P_-)\right)=\varepsilon$
($\varepsilon=0$ corresponds to $\hat{\rho}=\frac{1}{2}\hat{1}$, i.e. no information at all about state of "spin").

SINGLE-QUBIT OPERATIONS

It is helpful to think explicitly in terms of a literal spin-1/2 particle, e.g a nuclear spin: results are easily transcribed to the case of a general 2-state (qubit) system. Let's suppose we start with a fixed dc field $\underset{\sim}{\mathcal{H}}_0$ along the z-axis, and allow the spin to equilibrate, i.e. to reach state $|0\rangle$.

(Δ: implies (sufficiently weakly coupled) thermal bath with $kT \ll \mu\mathcal{H}_0$)



Apply oscillating rf field in xy-plane at Larmor frequency; ideally

$$\underset{\sim}{\mathcal{H}}_{rf} = \mathcal{H}_x \cos \omega_0 t + \mathcal{H}_y \sin \omega_0 t,$$
$$\omega_0 \equiv \gamma \mathcal{H}_0$$

In practice, usually use plane-polarized rf and neglect effects of "counter-rotating" component).

In frame rotating with Larmor frequency $\gamma\mathcal{H}_0$, $(\equiv \omega_0)$

$$\varphi = \text{const.(say 0)}, \ \theta = \gamma \int_0^t |\mathcal{H}_{rf}| dt \sim \gamma \mathcal{H}_{rf} \tau$$

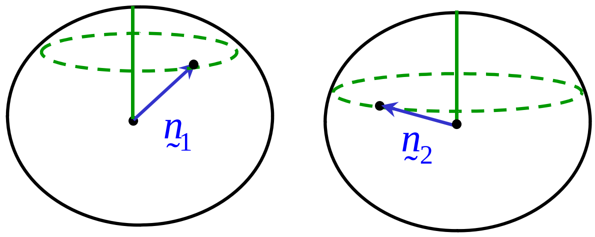
In lab. frame, θ is same but φ precesses at Larmor frequency $\gamma\mathcal{H}_0$. Can fix φ at any desired value (in principle!) by turning off $\underset{\sim}{\mathcal{H}}_0$. Thus, can access any target state.

Minimum time required (easiest to see with plane-polarized rf: must be able to neglect counter-rotating term):

$$\tau \gg \omega_0^{-1}$$

2-QUBIT OPERATIONS

A separable 2-qubit state can be represented by 2 Bloch spheres:



$$|\Psi\rangle = |\varphi(\underset{\sim}{n}_1)\rangle \, |\chi(\underset{\sim}{n}_2)\rangle$$

$$\varphi(\underset{\sim}{n}_1) = \cos\frac{\theta_1}{2}|0\rangle_1 + \sin\frac{\theta_1}{2} e^{i\varphi_1}|1\rangle_1, \text{ etc.}$$

However, an entangled 2-qubit state has no such simple pictorial representation: at best we can write

$$|\Psi\rangle = \iint d\underset{\sim}{n}_1 \, d\underset{\sim}{n}_2 \, \underbrace{\Psi(\underset{\sim}{n}_1,\underset{\sim}{n}_2)}_{\text{complex function}} |\varphi(\underset{\sim}{n}_1)\rangle \, |\chi(\underset{\sim}{n}_2)\rangle$$

or more simply

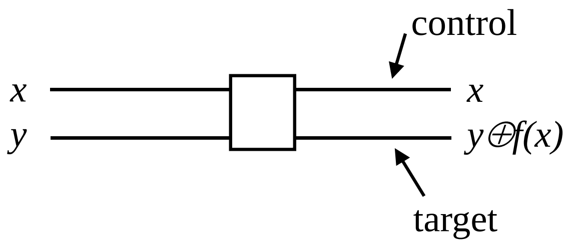
$$|\Psi\rangle = \alpha_{00}|0\rangle_1|0\rangle_2 + \alpha_{01}|0\rangle_1|0\rangle_2 + \alpha_{10}|1\rangle_1|0\rangle_2 + \alpha_{22}|0\rangle_2|0\rangle_2$$

How to create entangled states, starting from unentangled ones?

Ex: CNOT gate.

The classical CNOT gate is a special case of the gate

$$(x,y) \to \left(x, y \oplus f(x)\right)$$



for (e.g.) $f(0)=0, \quad f(1)=1.$

It has the action

$$\begin{aligned} (0,0) &\to (0,0) \\ (0,1) &\to (0,1) \\ (1,0) &\to (1,1) \\ (1,1) &\to (1,0) \end{aligned}$$

The corresponding quantum CNOT gate corresponds to a Û represented by the matrix

$$\hat{U}_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Suppose we start from some simple unentangled 2-qubit state, e.g.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1 + |1\rangle_1)|0\rangle_2$$

Then application of $\hat{U}_{CNOT}$ gives (since $|0\rangle_1|0\rangle_2 \to |0\rangle_1|0\rangle_2$,
$|1\rangle_1|0\rangle_2 \to |1\rangle_1|1\rangle_2$)

$$\hat{U}_{CNOT}|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2) \qquad (*)$$

which is clearly entangled. Note that $\hat{U}^{-1}_{CNOT}$ is well defined (actually $=\hat{U}_{CNOT}$) and "disentangles" the state (*).

It is a theorem (not proved here!)* that for a 2-qubit system the combination of all single-qubit unitary gates and CNOT is universal, i.e. starting from (e.g.) $|0\rangle_1|0\rangle_2$ we can reach any arbitrary target state of the 2-qubit system, entangled or not. Further, an arbitrary state of an N-qubit system, however entangled, can be reached by applying a succession of single-qubit gates and 2-qubit CNOTs. Thus,

single-qubit operations + CNOT suffices for QC.

How to Implement $\hat{U}_{CNOT}$ in Real-Life System?

Most convenient method may depend on specific physical nature of 2-qubit system, but consider for definiteness 2 real (e.g. nuclear) spins ½:

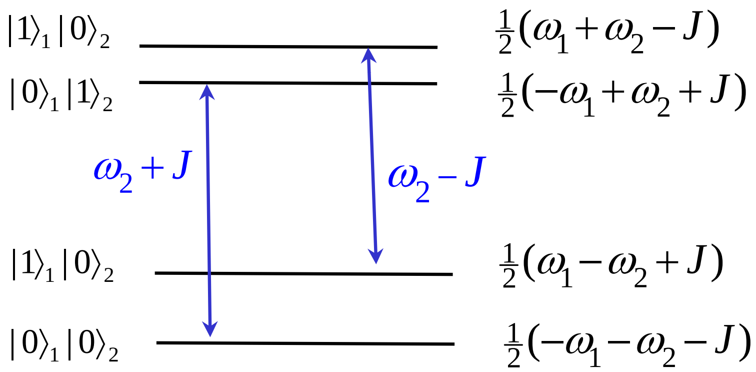
*See e.g. Nielsen and Chuang section 4.5.2.

2-SPIN QUANTUM CNOT GATE (schematic)

Consider 2 spin-1/2 nuclei with different magnetic moments in dc field $\mathcal{H}_0 \parallel \underset{\sim}{z}$, coupled by Ising-type interaction:

$$\hat{H} = -\frac{1}{2}\gamma_1 \sigma_{z1}\mathcal{H}_0 - \frac{1}{2}\gamma_2 \sigma_{z2}\mathcal{H}_0 - \frac{1}{2}J\sigma_{z1}\sigma_{z2} \qquad (\hbar = 1)$$

Energy level diagram:



Suppose we wish to treat 1 as the "control" and 2 as the "target" bit. Then $\hat{U}_{CNOT}$ should leave the states $|0\rangle_1|0\rangle_2$ and $|0\rangle_1|1\rangle_2$ unaffected but interchange $|1\rangle_1|0\rangle_2$ and $|1\rangle_1|1\rangle_2$. This can be done by applying an rf field with polarization along x-axis (i.e. plane-polarized) with frequency $\approx \omega_2 - J$,

$$\mathcal{H}(t) = \hat{\underset{\sim}{x}}\mathcal{H}_{rf}\ \underbrace{f(t)}_{\text{slowly varying envelope function}} \cos(\omega_2 - J)t$$

such that

$$2\gamma\mathcal{H}_{rf}\int_{-\infty}^{\infty} f(t)\,dt \approx \pi.$$

This will not induce transitions $|0\rangle_1|0\rangle_2 \leftrightarrow |0\rangle_1|1\rangle_2$ provided the Fourier component of $\mathcal{H}(t)$ at $\omega_2 + J$ is negligible, which roughly speaking implies (for "reasonable" (f(0)).

$$\underbrace{T}_{\text{“length” of pulse}} \gg J^{-1}$$

(so, conditions may be more stringent than for 1-qubit gate)

SOME PROBLEMS IN IMPLEMENTING QC IN REAL LIFE

1. Initialization: must be able to be sure of starting from a known initial state (typically $|0\rangle_1|0\rangle_2\ldots|0\rangle_N$)
2. Readout: must be able to read out final state (typically an eigenstate of computational basis such as $|0\rangle_1|1\rangle_2|1\rangle_3\ldots|0\rangle_N$) with high fidelity.
3. Reliability of 1- and 2-qubit operations.
4. Individual addressability of qubits (e.g. by spatial/frequency selection)
5. Scalability (crudely speaking, difficulty of implementing QC with N qubits should be $\sim N^d$ not e^N)

Both microscopic and macroscopic (superconducting) qubits meet these requirements to varying degrees. But the most generic and often the most severe problem in QC is

DECOHERENCE

Generic definition of decoherence: any deviation of actual quantum state from "target" one due to its interaction with its environment.

Suppose in a given basis target state is

$$\alpha_t|0\rangle+\beta|1\rangle \equiv \cos\tfrac{\theta_t}{2}|0\rangle+\sin\tfrac{\theta_t}{2}e^{i\varphi_t}|1\rangle$$

then, 2 kinds of error:

(a) $|\alpha| \neq \alpha_t$, i.e. $\theta \neq \theta_t$ ("bit flip")

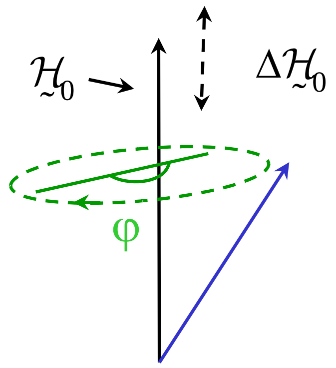
(b) $\arg(\alpha\beta^*) \neq \arg(\alpha_t\beta_t^*)$, i.e. $\varphi \neq \varphi_t$ ("phase flip")

but this is basis-dependent, e.g. phase flip in $\hat{\sigma}_z$-basis becomes bit flip in $\hat{\sigma}_x$-basis. Generally, define in computational basis.

ORIGINS OF DECOHERENCE

A. Environment describable classically (e.g. 50 Hz background, passing truck . . .)

Environment may exert random force on system. Simple example: spin in field constant in ($\underset{\sim}{z}$-) direction, but subject to random fluctuations in magnitude:



$$\underset{\sim}{\mathcal{H}}(\mathrm{t}) = \underset{\sim}{z}(\mathcal{H}_o + \underbrace{\Delta\mathcal{H}(\mathrm{t})}_{\text{random}})$$

$$\frac{d\underset{\sim}{n}}{dt} = \gamma\underset{\sim}{n}\times\underset{\sim}{\mathcal{H}}(\mathrm{t}) \Rightarrow \frac{d\varphi}{dt} = \gamma\mathcal{H}(\mathrm{t})$$

$\Rightarrow$ in Larmor frame, $\frac{d\varphi}{dt} = \gamma\Delta\mathcal{H}(\mathrm{t})$

$\Rightarrow$ in Larmor frame, $\varphi(\mathrm{t}) = \gamma\int_0^{\mathrm{t}}\Delta\mathcal{H}(\mathrm{t}')\mathrm{dt}'$

so for any one run ("realization"), φ(t) is definite, and if we could write a "density matrix for this realization" $\hat{\rho}\{\Delta\mathcal{H}(t)\}$, it would be

$$\hat{\rho}\{\Delta\mathcal{H}(t)\} = \begin{pmatrix} \cos^2\frac{\theta}{2}\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\ \exp i\varphi(t) \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2}\exp -i\varphi(t) & \sin^2\theta/2 \end{pmatrix}$$

But since we don't in practice know $\Delta\mathcal{H}$(t) and thus φ(t), we have to average over realizations of the noise, i.e. the "true" $\hat{\rho}$ is given by

$$\hat{\rho}(t) = \begin{pmatrix} \cos^2\frac{\theta}{2} & \sin\frac{\theta}{2}\cos\frac{\theta}{2}\ F(t) \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2}\,F^*(t) & \sin^2\frac{\theta}{2} \end{pmatrix}$$

where

$$F(t) \equiv \overline{\exp\ i\varphi(t)} \equiv \exp i\gamma\int_0^t \Delta\mathcal{H}(t')dt'$$

Recap: in presence of "longitudinal" classical noise diagonal elements of density matrix unaffected, but off-diagonal elements suppressed by factor

$$F(t) = \exp i\gamma \int_0^t \Delta\mathcal{H}(t')dt'$$

For Gaussian-distributed noise, $\overline{\exp i\alpha} = \exp -\frac{1}{2}\overline{\alpha^2}$, so

$$F(t) = \exp -\frac{1}{2}\gamma^2 \int_0^t \int_0^t dt'\, dt''\, \overline{\Delta\mathcal{H}(t')\Delta\mathcal{H}(t'')}$$

If noise is "white" $\left(\overline{\Delta\mathcal{H}(t')\Delta\mathcal{H}(t'')} = \eta\delta(t'-t'')\right)$

then $F(t) = \exp -\frac{1}{2}\gamma^2\eta t$

$\Rightarrow$ off-diagonal elements of $\hat{\rho}$ progressively suppressed to zero.

More generally, if

$$\overline{\Delta\mathcal{H}(t')\Delta\mathcal{H}(t'')} = (\Delta\mathcal{H})^2 f(t'-t''), \quad \int_{-\infty}^{\infty} f(x)dx = \tau$$

then for t»τ, $F(t) = \exp -\left\{\frac{1}{2}\gamma^2\left(\Delta\mathcal{H}\right)^2 \tau\right\} t$

so exponential suppression is fairly generic (though not universal).

B. Environment quantum-mechanical.

Suppose that we start with a separated state of system (S) and environment (E):

Set of coordinates of E

$$|\Psi_{S-E}\rangle = \left(\alpha|0\rangle + |\beta|1\rangle\right) \times \chi_{orig}(y)$$

so the system is in a pure state, with a density matrix

$$\hat{\rho}_S = \begin{pmatrix} |\alpha|^2 & \alpha^*\beta \\ \alpha\beta^* & |\beta|^2 \end{pmatrix}$$

Now suppose that as a result of S-E interaction, the S-E complex evolves so that $|\Psi'_{S-E}\rangle = \alpha|0\rangle|\chi_0(y)\rangle + \beta|1\rangle\chi_1(y)\rangle$

where $\langle\chi_0(y)|\chi_1(y)\rangle \neq 1$ (and may even be 0). What is $\hat{\rho}'_S$?

FUNDAMENTAL THEOREM OF DECOHERENCE:

If $\Psi_{S-E} = \alpha|0\rangle\chi_0(y)\rangle + \beta|1\rangle\chi_1(y)\rangle$, then the reduced density matrix $\hat{\rho}_S$ of S is given by

$$\hat{\rho}_S = \begin{pmatrix} |\alpha|^2 & \alpha^*\beta F \\ \alpha\beta^* F^* & |\beta|^2 \end{pmatrix} \qquad [\hat{\rho}_S = Tr_E\, \hat{\rho}_{S-E}]$$

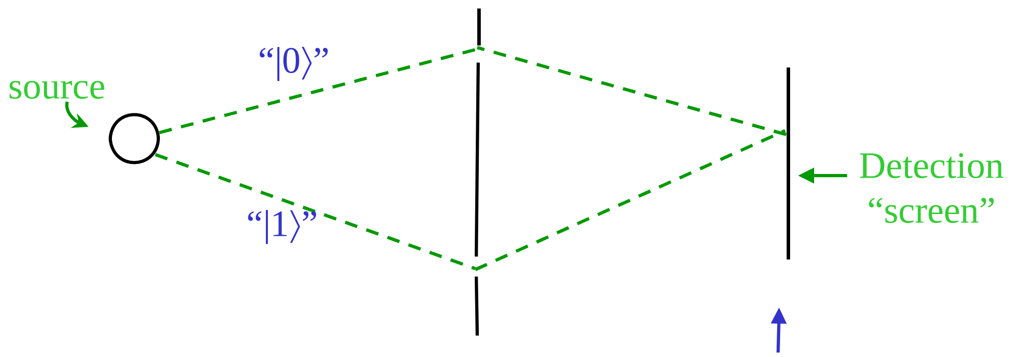
where

$$F \equiv \left\langle \chi_0(y) | \chi_1(y) \right\rangle \qquad (\text{so } 0 \le |F| \le 1)$$

Generally, F=F(t) and decreases with increasing t, usually (though not always) to 0 as $t \to \infty$.

Thus, at any given time t, effect of interaction with environment is qualitatively similar to that of classical noise.

Δ: However, unlike classical noise, the effect of correlations with a quantum environment may be "spontaneously reversible" ("false decoherence"): e.g. neutron in interferometer, effect of coupling of magnetic movement to zero-point radiation field!



$\Psi_{S-E} = (\alpha|0\rangle + \beta|1\rangle)|\chi_{or}(y)\rangle$

$\Psi_{S-E} = (\alpha|0\rangle + \beta|1\rangle) \cdot \chi_{or}(y)$

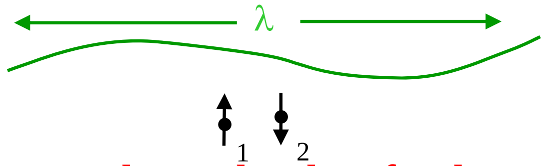
$\Psi_{S-E} = \alpha|0\rangle|\chi_0(y)$
$+\beta|1\rangle|\chi_1(y)\rangle,$
$\langle\chi_1(y)|\chi_1(y)\rangle \ll 1$

$\Rightarrow$ decoherence sometimes looks worse than it really is!

WAYS TO FIGHT DECOHERENCE:

1. Isolate system from its environment, while leaving it open to influence from experimenter. (ex: trapped ions)

2. Use "decoherence-free subspace" (DFS) (subspace in which coupling to environment is proportional to unit matrix). Ex: nuclear spins in solid, environment = long-wavelength blackbody field



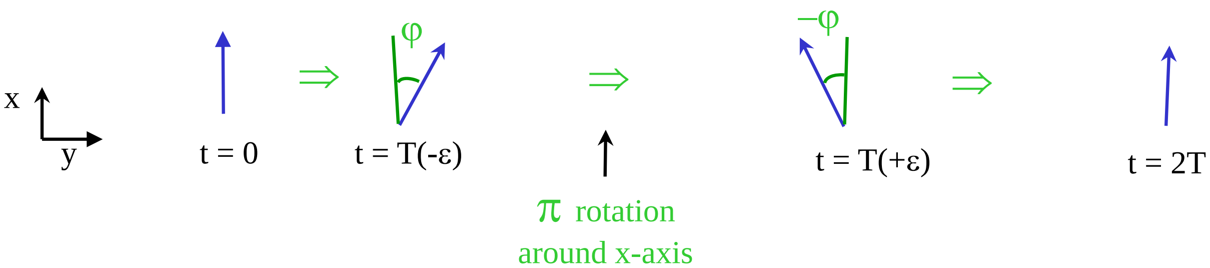
$$\hat{H}_{S-E} = \gamma(\hat{\sigma}_{1z} + \hat{\sigma}_{2z})\mathcal{H}(t)$$

$\Rightarrow$ states $|0\rangle_1 |1\rangle_2$, $|1\rangle_1 |0\rangle_2$ degenerate independently of value of $\mathcal{H}(t)$

$\Rightarrow$form DFS. (can use as single "logical" qubit)

3. (extreme form of (2)): use topologically protected subspace (guarantees any "local" operator $\propto \hat{1}$) (quantum Hall effect, Sr_2RuO_4. . . . —speculative at present)

4. Spin-echo ("bang-bang," "dynamical decoupling") technique, view down $\underset{\sim}{z}$-axis for "longitudinal" noise:



works provided

$$\int_0^{\tau} \mathcal{H}(t)dt = \int_{\tau}^{2\tau} \mathcal{H}(t)dt$$

i.e. provided correlation time of noise »T. (e.g. 1/f noise.

Ways to Fight Decoherence (cont.):

5. Quantum error correction

In most situations, the most dangerous type of decoherence is "phase noise," that is unwanted rotations of the relative phase of the states $|0\rangle$ and $|1\rangle$ (unwanted rotation of the "spin" around the z-axis in the computational basis). This will be generated by an interaction with the environment proportional to $\hat{Z}(\equiv\hat{\sigma}_Z)$: e.g. $\gamma\hbar\sigma_Z\mathcal{H}(t)$. But by an appropriate alternative choice of axes we can convert this to something proportional to $\hat{x}\left(\equiv\hat{\sigma}_x\equiv\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)$. This kind of error clearly interchanges the states in the new basis, so that

$$\alpha|0\rangle+\beta|1\rangle\Rightarrow\alpha|1\rangle+\beta|0\rangle.$$

Is it possible to detect and correct such errors? The problem is that any measurement on the qubit by itself is liable to destroy complete information about its state.

Let's bring in two more "ancilla" qubits and encode the state $\alpha|0\rangle_1 + \beta|\rangle_1$ as follows:

$$\alpha|0\rangle_1+\beta|1\rangle_1\Rightarrow\alpha|0_1 0_2 0_3\rangle+\beta|1_1 1_2 1_3\rangle$$

(We don't specify how we do this, but note that it does not violate the "no-cloning" theorem). Suppose now a single error occurs on one (and only one) of the three qubits (the probability of more than one error is $3p^2–2p^3$, which is $<p$ for $p <1/2$). We then have (including the original error-free state) the possibilities:

$$
\begin{array}{ll}
\alpha|0_1 0_2 0_3\rangle+\beta|1_1 1_2 1_3\rangle & \text{(no error)}\\
\alpha|1_1 0_2 0_3\rangle+\beta|0_1 1_2 1_3\rangle & \text{(error on bit 1)}\\
\alpha|0_1 1_2 0_3\rangle+\beta|1_1 0_2 1_3\rangle & \text{(error on bit 2)}\\
\alpha|0_1 0_2 1_3\rangle+\beta|1_1 1_2 0_3\rangle & \text{(error on bit 3)}
\end{array}
$$

Note that no states are common to any two rows.

Recap:

if no error, $\alpha|0_1 0_2 0_3\rangle + \beta|1_1 1_2 1_3\rangle$
if error on bit 1, $\alpha|1_1 0_2 0_3\rangle + \beta|0_1 1_2 1_3\rangle$
if error on bit 2, $\alpha|0_1 1_2 0_3\rangle + \beta|1_1 0_2 1_3\rangle$
if error on bit 3, $\alpha|0_1 0_2 1_3\rangle + \beta|1_1 1_2 0_3\rangle$

so each case falls in a different 2D subspace of the 3D Hilbert space!

(Note this doesn't work for 2 qubits, where one has 3 possibilities (no error, error on 1, error on 2) but only a 4D Hilbert space).

So if we make a measurement only of which of the 4 2D subspaces the system is in (i.e. we measure the projectors on these subspaces), without asking (e.g.) "are you $|0_1 0_2 0_3\rangle$ or $|0_1 1_1 1_3\rangle$?", then the subspace is "decoherence-free" as regards the measurement! So, supposing e.g. that the result is subspace 2 (error on bit 1), even after the measurement the state is still $\alpha|1_0 0_2 0_3\rangle + \beta|0_1 1_2 0_3\rangle$. Knowing the error, we now correct simply by applying the operator $\hat{X}_1 \hat{1}_2 \hat{1}_3$ (i.e. flip bit 1, leave bits 2 and 3 alone).

Δ: further developments necessary to deal with the possibility of bit flips as well as phase flips (in the computational basis). But ∃ many theorems are the effect that "fault-tolerant quantum computation" is possible provided the original probability of error is low enough. Usual figure of merit is the product $Q = \omega_0 T_\varphi$ where ω_0 is the inverse of the minimum time for a single 1- or 2-qubit gate (e.g. Larmor frequency for 1-qubit, J^{-1} for 2-qubit) and T_φ is the "decoherence time" (~ time for off-diagonal elements of $\hat{\rho}$ to →0) Usually thought that a minimum requirement is

$$Q \gtrsim 10^4$$