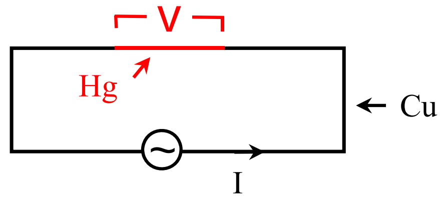
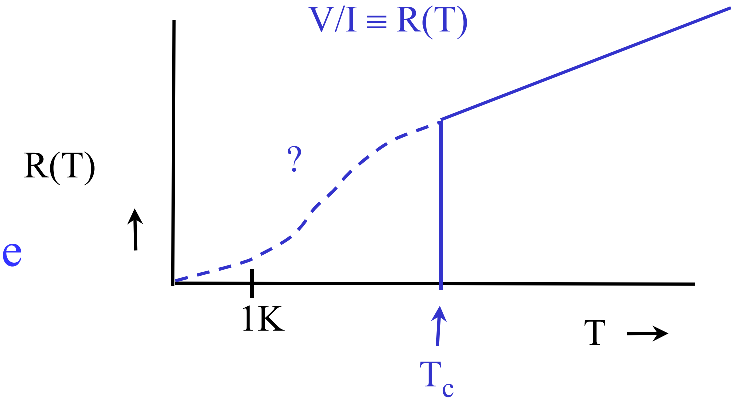
SUPERCONDUCTIVITY.

1. History:



Kamerlingh Onnes
(1911, Hg):

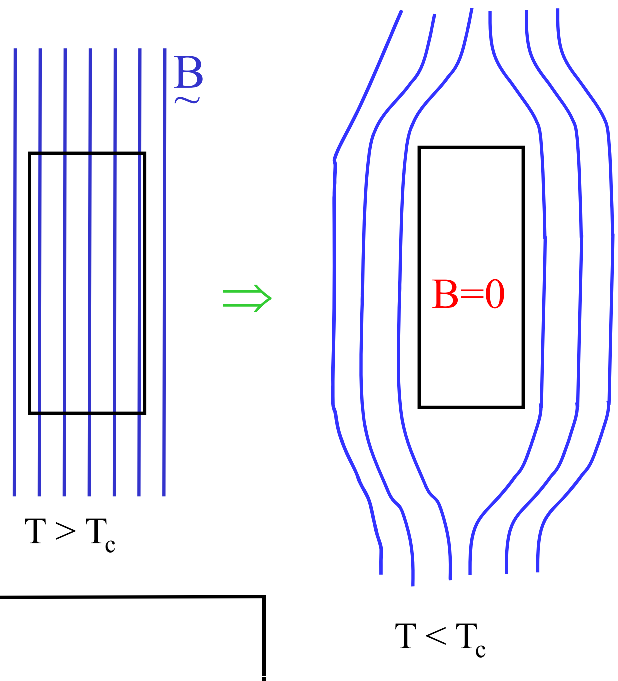
zero voltage drop



superconductivity ≡ absence of resistance?

theoretical attempts 1911–1933

Meissner and Ochsenfeld (1933):
total expulsion of flux
(equilibrium effect)



Londons 1935
isotope effect 1950
Ginzburg-Landau 1950

BCS theory (microscopic) 1957
Josephson effect 1962
["exotic" superconductivity 1975, high-temperature (cuprate) superconductivity 1986]
In these lectures, "superconductivity" always "classic" (BCS)

SUPERCONDUCTIVITY (cont.)

2. Phenomenology (classic, type-I)

Superconductivity sets in abruptly at temperature T_c (typically 1–20 K). Below T_c, superconducting state differs qualitatively from normal ($T>T_c$) state in 3 respects:

1. dc conductivity $\to \infty$ (e.g. persistent currents in ring)
2. magnetic flux completely expelled (Meissner effect)
3. Peltier coefficient $\to 0$

Occurrence:

metals, alloys, semiconductors
in metals, more towards middle of periodic table
⚠: "best" metals (Cu, Ag, Au) not superconductors
not sensitive to nonmagnetic impurities (e.g. many very "dirty" alloys good superconductors with $T_c \sim 20$K), very sensitive to magnetic impurities.

Normal state ($T>T_c$) of superconducting metal essentially a "textbook" metal described by Sommerfeld-Bloch-Landau theory

Isotope effect: for a given (elemental) metal

$$T_c \propto M^{-1/2} \qquad \text{isotopic mass}$$

$\Rightarrow$ Dynamics of nuclei (i.e. phonons) must play a role

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Microscopic properties in superconducting state:

Relative to extrapolated N-state values

specific heat	C_v	
spin susceptibility	χ	
ultrasound attenuation	α	$\Rightarrow 0$ for $T \to 0$
thermal conductivity	K	
nuclear relaxation rate	T^{-1}	

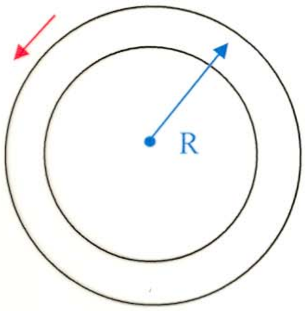
SUPERFLUIDITY IN LIQUID ^{4}HE

^{4}He liquefied:	1908	
$T < T_\lambda$ (2.17 K):	1920	~ 20 YEARS!
Frictionless flow below T_λ:	1938	

Modern point of view:

Define

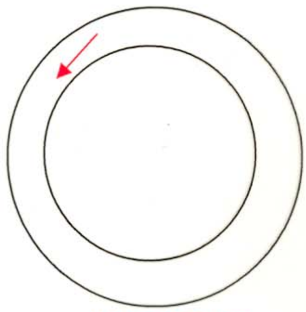
$\omega_c \equiv \hbar/mR^2$ $\equiv$ quantum unit of rotation ($\sim 10^{-4}$ Hz for $R \sim 1$cm)



EXPT. A
("Hess-Fairbank" effect)

walls rotate with
ang. velocity $\lesssim \omega_c$,
liquid stationary

EQUILIBRIUM
EFFECT



EXPT B
(Persistent currents)

walls at rest,
liquid rotating with
ang. velocity $\gg \omega_c$.

METASTABLE
EFFECT

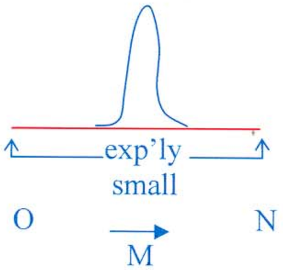
BEC IN A NONINTERACTING BOSE GAS: THE EFFECTS OF STATISTICS

I. Qualitative argument:

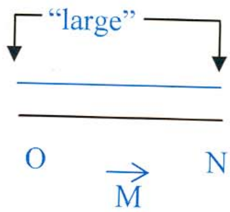
Distribute N objects between 2 boxes: what is probability P(M) of finding M in one box?

A. Objects distinguishable ($\equiv$ coin toss):

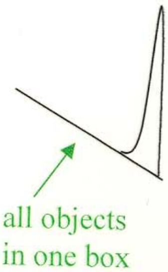
$$P(M) = \frac{N!}{M!\, N-M!}$$



B. Objects indistinguishable (bosons):



P(M) = 1/N



II. Quantitative argt. (Einstein, 1925):

chemical potential, ≤ 0

$$n_i(T) = [\exp(\varepsilon_i - \mu(T)/k_B T - 1)]^{-1}$$

$\mu(T)$ fixed by: $\sum_i n_i(T{:}\ \mu(T)) = N$ ← total no. of particles

$T \to \infty \Rightarrow \mu \to -\infty$: $T\downarrow \Rightarrow \mu\uparrow$. But what if

$$\sum_i [\exp(\varepsilon_i/k_B T) - 1]^{-1} < N?$$

Einstein: Macroscopic no. of particles occupy lowest ($\varepsilon = 0$) state!

BEC IN A GENERAL (INTERACTING, NONEQUILIBRIUM) SYSTEM:

Can always find set of "single-particle" states $\chi_i(\mathbf{r},t)$ st. average no. of atoms in state i is $n_i(t)$ (and $\langle a_i^+ a_j\rangle \equiv 0,\ i \neq j$)

Df of ("simple") BEC:

one and only one single-particle state
i (say i = 0) has $n_i(t) = 0(N)$, rest $o(1)$

Then,

$N_0(t) \equiv$ "condensate number"
$\chi_0(\mathbf{r},t) \equiv$ "condensate wave function"

WHY BEC IN GENERAL CASE?

A. Statistics
B. Interactions ("Fock" term):
e.g. if $V(\mathbf{r}) = V_0\, \delta(\mathbf{r})$:

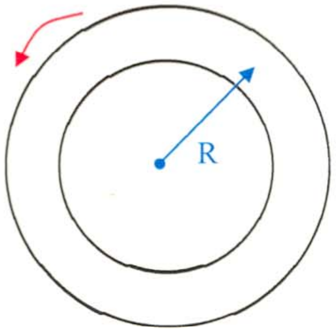
N atoms in $\chi_0(rt)$: $\langle V\rangle(t) = \frac{1}{2} V_0 N^2 \cdot \int |\chi_0(\mathbf{r}t)|^4\, dr$

N_1 in χ_1, N_2 in χ_2: $\langle V\rangle(t) = 2V_0 N_1 N_2 \int |\chi_1(\mathbf{r}t)|^2 |\chi_2(\mathbf{r}t)|^2\, dr$

$\Rightarrow$ if $V_0 > 0$, advantageous to have all in one state

what if $V_0 < 0$?

EXPLANATION OF HESS-FAIRBANK EFFECT IN TERMS OF BEC:



Walls rotating with ang. velocity

$\omega \lesssim \omega_c \Leftarrow \equiv \hbar/m R^2$

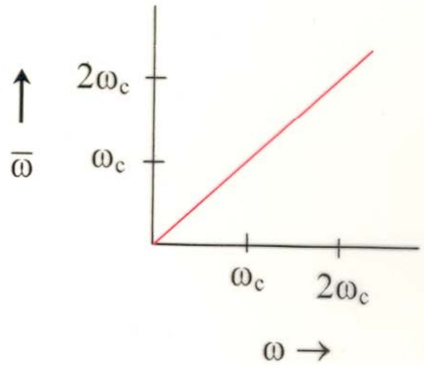
What does liquid do?

General principle: Average ang. velocity of atoms ($\bar{\omega}$) as close as possible to ω

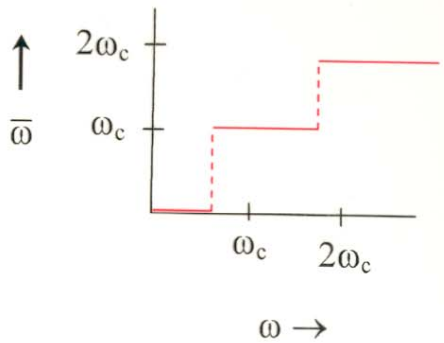
↑ : Single-atom states must obey quantization condition: $\omega = n\omega_c$ ($\ell = n\hbar$)

A. "Normal" (non-BEC) system: many different single-particle states occupied (typical value of $n \sim (kT/\hbar\omega_c)^{1/2} \sim 10^7$)

$\Rightarrow$ to get $\bar{\omega} = \omega$, just shift atoms slightly between states.



B. BEC system ($T \ll T_c$) (almost) all atoms in condensate $\rightarrow$ must have same value of n (n_o) $\Rightarrow \bar{\omega} \cong n_o\,\omega_c$



INTERACTIONS "OPTIONAL"

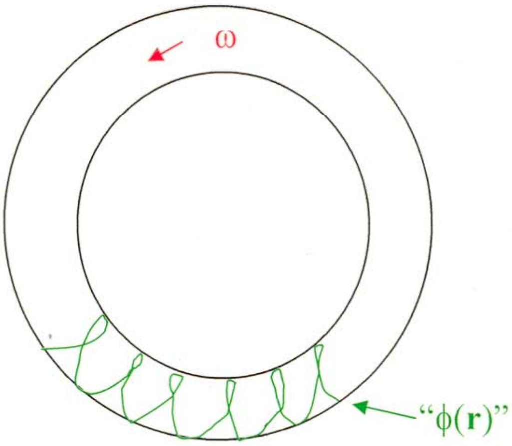
^{4}He: PERSISTENT CURRENTS

Initially, after walls stopped,

$\langle L \rangle = N_o \ell_o \hbar, \quad \ell_o \gg 1 \quad (\bar{\omega} \gg \omega_c)$

But groundstate has $\langle L \rangle = 0$. $(\omega = 0)$

Why no relaxation?



$\chi_o(\mathbf{r}) = |\chi_o(\mathbf{r})| \exp i\, \phi(\mathbf{r})$

condensate w.f.

Df: "winding no." $n \equiv \oint \dfrac{\nabla\phi \cdot d\mathbf{l}}{2\pi}$

Initially, $n = \ell_o$: eq^m state has $n = 0$.

To change n, must depress $|\chi_o(\mathbf{r})|$ to zero somewhere!

(a) Electron in atom:

Schrödinger eqn. linear ⇒ nodes cost no extra energy, e.g.

$\psi(t) = a(t)\, \psi_p + b(t)\, \psi_s \qquad \begin{cases} t \to -\infty: & a=1,\ b=0 \\ t \to +\infty: & a=0,\ b=1 \end{cases}$

$\langle E \rangle(t) = |a(t)|^2 E_p + |b(t)|^2 E_s$ = monotonically decreasing

(b) BEC (^{4}He):

Extra term in energy: $\langle V \rangle = \underset{>0}{V_o} \int |\chi_o(\mathbf{r}t)|^4 \, d\mathbf{r}$

⇒ energy NOT monotonically decreasing!

(REPULSIVE) INTERACTIONS ESSENTIAL!

Correspondence between Superfluidity and Superconductivity

a) Persistent currents in ^{4}He in annular geometry $\Leftrightarrow$ persistent currents in superconducting ring.

b) What is superconducting analog of Hess-Fairbank effect?

   i. Behavior of superconductor under rotation ("London moment")

   ii. less obviously: behavior in magnetic field

Neutral system observed from rotating frame

$$\hat{H} = \hat{H}_0 - \underset{\sim}{\omega} \cdot \underset{\sim}{\hat{L}}$$
$$\equiv \frac{1}{2m}\sum_i (p_i - m\underset{\sim}{\omega} \times \underset{\sim}{r}_i)^2 + \sum_i V(r_i)$$
$$+ \frac{1}{2}\sum_{ij} U(\underset{\sim}{r}_i - \underset{\sim}{r}_j)$$
$$- \frac{1}{2}\sum_i m(\underset{\sim}{\omega} \times \underset{\sim}{r}_i)^2 \leftarrow \text{centrifugal term, affects only meniscus}$$

Charged system in magnetic field, observed from lab frame

$$\hat{H} = \hat{H}_0 - e\sum_i p_i \cdot \underset{\sim}{A}(r_i) + \frac{1}{2}\sum_i \frac{e^2 A^2}{m}(r_i)$$
$$\equiv \sum_i (p_i - \tfrac{e}{2}\underset{\sim}{B} \times \underset{\sim}{r}_i)^2 + \sum_i V(r_i)$$
$$+ \frac{1}{2}\sum_{ij} U(r_i - \underset{\sim}{r}_j)$$

So:

neutral system observed in container rotating with velocity $\underset{\sim}{\omega}$ viewed from rotating frame $\Leftrightarrow$ charged system in magnetic field B, viewed from lab frame

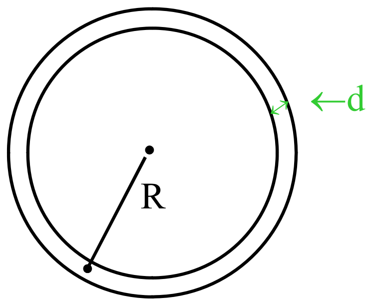
(with "scaling" $\underset{\sim}{B} \rightleftarrows \frac{e}{2m}\underset{\sim}{\omega}$)

$\Downarrow$

HF effect: (part of) system at rest in lab. frame $\Rightarrow$ moving in rotating frame $\Leftrightarrow$ (part of) system moving in lab frame (diamagnetism)

SUPERFLUID-SUPERCONDUCTOR CORRESPONDENCE (cont.)

Quantitative correspondence (T=0): consider in each case "thin" ring (d«R) then $\underset{\sim}{v} \equiv \underset{\sim}{\omega} \times \underset{\sim}{r} \cong \underset{\sim}{\omega} \times \underset{\sim}{R} \equiv R\underset{\sim}{\omega} \times \underset{\sim}{\hat{n}} \neq f(r)$ and (prima facie!) $\underset{\sim}{A}(\underset{\sim}{r}) \equiv \frac{1}{2}\underset{\sim}{B} \times \underset{\sim}{r} = \frac{1}{2}R\underset{\sim}{B} \times \underset{\sim}{\hat{n}} \neq f(r)$



Neutral case (T=0): as viewed from rotating frame,

mass current density $\rightarrow \underset{\sim}{J} = -n\, m\, \underset{\sim}{v}$

$\uparrow$ particle density

So in charged case $\underset{\sim}{J}_{el} = -\frac{ne^2}{m}\underset{\sim}{A}$

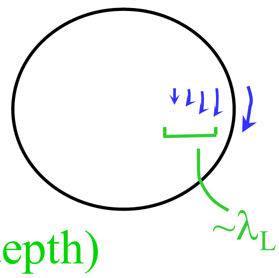
electric current density — vector potential

On a "sufficiently coarse-grained" scale, can interpret this as a local relation between $\underset{\sim}{J}_{el}$ and $\underset{\sim}{A}$:

$$\underset{\sim}{J}_{el}(\underset{\sim}{r}) = -\frac{ne^2}{m}\underset{\sim}{A}(\underset{\sim}{r}) \qquad \text{(London equation)}$$

But, in a bulk geometry, $\underset{\sim}{A}(\underset{\sim}{r})$ must be determined self-consistently from Maxwell's equations, which in time-independent case ⇒ $\nabla^2\underset{\sim}{A}(\underset{\sim}{r}) = -J_{el}(r)/\varepsilon_0$. Hence,

$$\left.\begin{array}{l} \nabla^2\underset{\sim}{A}(\underset{\sim}{r}) = \lambda_L^{-2}\underset{\sim}{A}(\underset{\sim}{r}) \\ \Rightarrow \nabla^2\underset{\sim}{B}(\underset{\sim}{r}) = \lambda_L^{-2}\underset{\sim}{B}(\underset{\sim}{r}) \end{array}\right\} \quad \begin{array}{l} \lambda_L \equiv \left(\dfrac{m\varepsilon_0}{ne^2}\right)^{-1/2} \\ \equiv c/\omega_p \end{array}$$



and $\underset{\sim}{B}(\underset{\sim}{r})$ falls off (London penetration depth) exponentially in bulk superconductor over distance $\sim\lambda_L$ ⇒ Meissner effect.

⚠: 1) In multiply connected superconductor, London equation must be generalized (but $\underset{\sim}{B}(\underset{\sim}{r})$ still falls off exponentially)

2) London equation not quantitatively valid in type-I superconductors (∵ not "sufficiently coarse-grained")

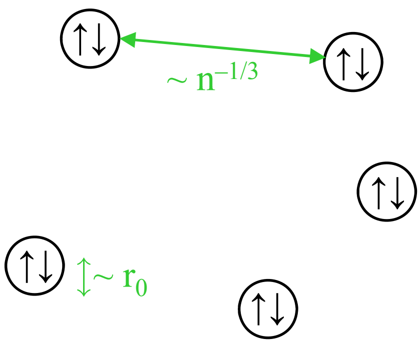
How to Adapt (Qualitative) Ideas re BEC to Superconductivity? (⚠: non-historical!)

Obvious problem in taking over idea of BEC directly: electrons in superconductor are fermions! So,

$$\rho_1(\mathbf{r},\mathbf{r}') = \sum_i n_i \chi_i^*(r)\chi_i(r'), \qquad n_i \leq 1 \qquad \text{(Pauli principle)}$$

$\Longrightarrow$ no BEC in literal sense. However: consider hypothetical dilute gas of diatomic molecules composed of 2 fermions (for simplicity with ℓ=S=0), with $nr_0^3 \ll 1$

density molecular radius



(Ex: hypothetical gaseous D_2)

It is highly plausible that in the limit $nr_0^3 \to 0$, this system will behave just like a dilute gas of bosons (with spin 0). Moreover, while the details of the molecule-molecule interactions depend on the short-range part of the potential, at least in the limit of "large" molecules there are strong arguments* that it should be repulsive. Thus the model is exactly that discussed above, and in the limit T→0 we expect

BEC of Di-Fermionic Molecules

What does the many-body wave function of such a system look like? Answer:

antisymmetrizer BEC!

$$\Psi(\mathbf{r}_1\sigma_1\mathbf{r}_2\sigma_2\ldots\mathbf{r}_N\sigma_N) = \mathcal{A}\varphi(\mathbf{r}_1\mathbf{r}_2\sigma_1\sigma_2)\varphi(\mathbf{r}_3\mathbf{r}_4\sigma_3\sigma_4)\ldots\varphi(\mathbf{r}_N-1,\mathbf{r}_N,\sigma_{N-1},\sigma_N)$$

molecular state

$$\varphi(\mathbf{r}_1\mathbf{r}_2\sigma_1\sigma_2) = \tfrac{1}{\sqrt{2}}(\uparrow_1\downarrow_2 - \downarrow_1\uparrow_2)\varphi(|\mathbf{r}_1-\mathbf{r}_2|)$$

spin singlet s-wave

*Petrov et al., PRL 93, 090404(2004): $a_{BB}=0.6a_{FF}(>0)$.

How to Adapt BEC Ideas. . .? (cont.)

2-body problem

Consider the behavior of two isolated fermions with some interatomic potential $V\left(|\underset{\sim}{r}_1 - \underset{\sim}{r}_2|\right)$ whose strength (and/or "shape") can be varied, in a $\underline{\mathrm{K}} = 0$, $\mathrm{S} = \ell = 0$ state

total momentum — spin — relative orbital angular momentum

As the potential is varied, its effects on the low-energy behavior are uniquely parametized by the quantity a_s (s-wave scattering length), or more conveniently by a_s^{-1} :

For a strongly attractive potential $a_s^{-1} \to +\infty$: fermions form tightly bound molecule (radius ~ range of potential (w)).

As potential is weakened, a_s^{-1} decreases and eventually becomes negative.

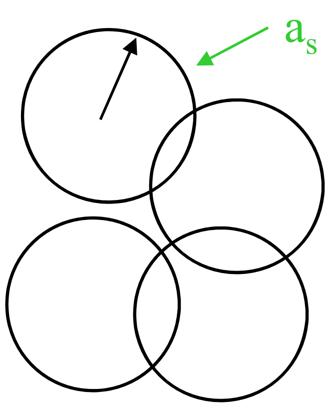
For $a_s^{-1} > 0$ but $\ll w^{-1}$, fermions form weakly bound molecule (radius = $a_s \gg w^{-1}$), with binding energy $\varepsilon = -\hbar^2 / m a_s^2$.

For $a_s^{-1} \to 0$ ("unitarity") the energy of the molecular bound state $\to 0$, and for $a_s^{-1} > 0$ no molecular state is possible.

Now, back to the many-body problem: What happens if starting from a dilute BEC of di-fermionic molecules, we gradually weaken the inter-fermion attraction (while keeping n = const.)?

When the (2-body) s-wave scattering length a_s becomes $\sim n^{-1/3}$, "molecules ¨start to overlap $\Rightarrow$ cannot neglect effects of Pauli principle. Equivalently,

$$"\varepsilon_F" \sim \frac{n^{2/3}\hbar^2}{m} \sim \frac{\hbar^2}{m a_s^2} \sim |E|.$$



When $a_s \gtrsim n^{-1/3}$, do "molecules" unbind?

DO MOLECULES UNBIND? (cont.)

BCS (1957): (A sort of) "molecules" persist for arbitrarily weak attraction, i.e. even for a_s -ve when no 2-body state is bound.

Unifying formalism (Yang, 1962):

General many-body pure state wave function:

$$\Psi_N^S(t) \equiv \Psi_S\left(\underset{\sim}{r}_1\sigma_1, \underset{\sim}{r}_2\sigma_2 \ldots \underset{\sim}{r}_N\sigma_N : t\right)$$

2-body density matrix ρ_2 defined by

prob. of status

$$\rho_2\left(r_1\sigma_1, r_2\sigma_2 : r_1'\sigma_1', r_2'\sigma_2'\right) \equiv N(N-1)\sum_S p_S \sum_{\sigma_3\sigma_4\ldots\sigma_{11}} \int d\underset{\sim}{r}_3 d\underset{\sim}{r}_4 \ldots d\underset{\sim}{r}_N$$

$$\Psi_S^*\left(\underset{\sim}{r}_1\sigma_1, \underset{\sim}{r}_2\sigma_2, \underbrace{\underset{\sim}{r}_3\sigma_3 \ldots \underset{\sim}{r}_N\sigma_N}\, : t\right) \bullet \Psi_S\left(\underset{\sim}{r}'_1\sigma_1', \underset{\sim}{r}'_2\sigma_2', \underbrace{\underset{\sim}{r}_3\sigma_3 \ldots \underset{\sim}{r}_N\sigma_N}\, : t\right)$$

summed/integrated over

$$\equiv \left\langle \psi_{\sigma_1}^\dagger\left(\underset{\sim}{r}_1 t\right)\psi_{\sigma_2}^\dagger\left(\underset{\sim}{r}_2 t\right)\psi_{\sigma_2'}\left(\underset{\sim}{r}_2' t\right)\psi_{\sigma_1'}\left(\underset{\sim}{r}_1' t\right)\right\rangle$$

$\hat{\rho}_2$ is Hermitian $\Rightarrow$ can be diagonalized:

$$\rho_2\left(\underset{\sim}{r}_1\sigma_1, \underset{\sim}{r}_2\sigma_2 : \underset{\sim}{r}_1'\sigma_1', \underset{\sim}{r}_2'\sigma_2' : t\right) = \sum_i n_i(t)\chi_i^*\left(\underset{\sim}{r}_1\sigma_1, \underset{\sim}{r}_2\sigma_2 : t\right)\chi_i\left(\underset{\sim}{r}_1'\sigma_1', \underset{\sim}{r}_2'\sigma_2 : t\right)$$

eigenvalue eigenfunction

Theorem (Yang): All $n_i \leq N$.

Ansatz: (In thermal equilibrium at T< some T_c): For arbitrarily weak attraction, $\exists$ one and only one eigenvalue ~ N, all others ~1. ("ODLRO"). Call it N_0, and corresponding χ χ_0

BEC limit $(a_S^{-1} \to +\infty)$: $N_0 = N, \chi_0 =$ molecular wave function

BCS limit $(a_S^{-1} \to -\infty)$: $N_0 \ll N$ (but ~ N), χ_0 "molecular-like" but radius $\gg n^{-1/3}$.

intermediate case ("unitarity") ?? ("BEC-BCS crossover" in Fermi alkali gases)

Qualitative Argument for Main Phenomena of Superconductivity from Yang Hypothesis

Bose case (recap):

$$\rho_1\left(\underset{\sim}{r}_1,\underset{\sim}{r}':t\right)=\sum_i n_i(t)\chi_i^*\left(\underset{\sim}{r},t\right)\chi_i(\underset{\sim}{r}'t)$$

in thermal equilibrium, $\exists$, one eigenvalue (N_0) ~N ("BEC"), with associated eigenfunction χ_0(r). Define order parameter

$$\Psi(r)\equiv\sqrt{N_0}\,\chi_0(\underset{\sim}{r}) \qquad \Psi(\underset{\sim}{r})\equiv|\Psi(\underset{\sim}{r})|\exp i\,\varphi(\underset{\sim}{r})$$

Then: a) Hess-Fairbank effect follows from BEC alone.

b) stability of supercurrents follows from BEC plus repulsive interactions, i.e. term in energy $\sim b\,|\Psi(r)|^4,\ b>0.$

Fermi case:

$$\rho_2\left(r_1\sigma_1,r_2\sigma_2:r_1'\sigma_1',\underset{\sim}{r}_2'\sigma_2':t\right)=\sum_i n_i(t)\chi_i^*\left(r_1\sigma_1,r_2\sigma_2:t\right)\chi_i\left(r_1'\sigma_1',\underset{\sim}{r}_2'\sigma_2:t\right)$$

Assumption: in thermal equilibrium $\exists$ one eigenvalue (N_0)~N, with associated eigenfunction $\chi_0\left(\underset{\sim}{r}_1\sigma_1,\underset{\sim}{r}_2\sigma_2\right)$.

Write

$$\chi_0\left(\underset{\sim}{r}_1\sigma_1,\underset{\sim}{r}_2\sigma_2\right)\equiv\chi_0\left(\underbrace{\underset{\sim}{r}_1-\underset{\sim}{r}_2}_{\text{relative coordinate}},\sigma_1\sigma_2:\underbrace{\underset{\sim}{r}}_{\text{COM coordinate}}\right)$$

and fix $\left(\underset{\sim}{r}_1-\underset{\sim}{r}_2,\sigma_1\sigma_2\right)$ at some "standard" values (e.g. for s-wave, $\left(\underset{\sim}{r}_1-\underset{\sim}{r}_2=0,\ \sigma_1=-\sigma_2=\uparrow\right)$.Then $\chi_0\equiv\chi_0(r)$, and can define similarly to Bose case an order parameter

$$\Psi(r)\equiv\sqrt{N_0}\,\chi_0(\underset{\sim}{r}) \qquad \Psi(\underset{\sim}{r})\equiv|\Psi(\underset{\sim}{r})|\exp i\,\varphi(\underset{\sim}{r})$$

So, arguments go through similarly to Bose case, provided $\exists$ a term in energy of form $\sim b\,|\Psi(r)|^4,\ b>0.$

(Note: $\Psi(\underset{\sim}{r})$ is essentially the order parameter introduced in the phenomenological theory of Ginzburg and Landau, without an appreciation of its microscopic meaning.)