THE
INTERNATIONAL SERIES
OF
MONOGRAPHS ON PHYSICS

International Series of Monographs on Physics

121. T. Giamarchi: *Quantum physics in one dimension*
120. M. Warner, E. Terentjev: *Liquid crystal elastomers*
119. L. Jacak, P. Sitko, K. Wieczorek, A. Wojs: *Quantum Hall systems*
118. J. Wesson: *Tokamaks, Third edition*
117. G. E. Volovik: *The universe in a helium droplet*
116. L. Pitaevskii, S. Stringari: *Bose–Einstein condensation*
115. G. Dissertori, I. G. Knowles, M. Schmelling: *Quantum chromodynamics*
114. B. DeWitt: *The global approach to quantum field theory*
113. J. Zinn-Justin: *Quantum field theory and critical phenomena, Fourth edition*
112. R. M. Mazo: *Brownian motion: fluctuations, dynamics, and applications*
111. H. Nishimori: *Statistical physics of spin glasses and information processing: an introduction*
110. N. B. Kopnin: *Theory of nonequilibrium superconductivity*
109. A. Aharoni: *Introduction to the theory of ferromagnetism, Second edition*
108. R. Dobbs: *Helium three*
107. R. Wigmans: *Calorimetry*
106. J. Kübler: *Theory of itinerant electron magnetism*
105. Y. Kuramoto, Y. Kitaoka: *Dynamics of heavy electrons*
104. D. Bardin, G. Passarino: *The standard model in the making*
103. G. C. Branco, L. Lavoura, J. P. Silva: *CP violation*
102. T. C. Choy: *Effective medium theory*
101. H. Araki: *Mathematical theory of quantum fields*
100. L. M. Pismen: *Vortices in nonlinear fields*
99. L. Mestel: *Stellar magnetism*
98. K. H. Bennemann: *Nonlinear optics in metals*
97. D. Salzmann: *Atomic physics in hot plamas*
96. M. Brambilla: *Kinetic theory of plasma waves*
95. M. Wakatani: *Stellarator and heliotron devices*
94. S. Chikazumi: *Physics of ferromagnetism*
91. R. A. Bertlmann: *Anomalies in quantum field theory*
90. P. K. Gosh: *Ion traps*
89. E. Simánek: *Inhomogeneous superconductors*
88. S. L. Adler: *Quaternionic quantum mechanics and quantum fields*
87. P. S. Joshi: *Global aspects in gravitation and cosmology*
86. E. R. Pike, S. Sarkar: *The quantum theory of radiation*
84. V. Z. Kresin, H. Morawitz, S. A. Wolf: *Mechanisms of conventional and high* T_c *superconductivity*
83. P. G. de Gennes, J. Prost: *The physics of liquid crystals*
82. B. H. Bransden, M. R. C. McDowell: *Charge exchange and the theory of ion–atom collisions*
81. J. Jensen, A. R. Mackintosh: *Rare earth magnetism*
80. R. Gastmans, T. T. Wu: *The ubiquitous photon*
79. P. Luchini, H. Motz: *Undulators and free-electron lasers*
78. P. Weinberger: *Electron scattering theory*
76. H. Aoki, H. Kamimura: *The physics of interacting electrons in disordered systems*
75. J. D. Lawson: *The physics of charged particle beams*
73. M. Doi, S. F. Edwards: *The theory of polymer dynamics*
71. E. L. Wolf: *Principles of electron tunneling spectroscopy*
70. H. K. Henisch: *Semiconductor contacts*
69. S. Chandrasekhar: *The mathematical theory of black holes*
68. G. R. Satchler: *Direct nuclear reactions*
51. C. Møller: *The theory of relativity*
46. H. E. Stanley: *Introduction to phase transitions and critical phenomena*
32. A. Abragam: *Principles of nuclear magnetism*
27. P. A. M. Dirac: *Principles of quantum mechanics*
23. R. E. Peierls: *Quantum theory of solids*

Quantum Physics in One Dimension

THIERRY GIAMARCHI
University of Geneva
and
Paris XI University

CLARENDON PRESS • OXFORD
2003

OXFORD
UNIVERSITY PRESS

Great Clarendon Street, Oxford OX2 6DP

Oxford University Press is a department of the University of Oxford.
It furthers the University's objective of excellence in research, scholarship,
and education by publishing worldwide in

Oxford New York

Auckland Bangkok Buenos Aires Cape Town Chennai
Dar es Salaam Delhi Hong Kong Istanbul Karachi Kolkata
Kuala Lumpur Madrid Melbourne Mexico City Mumbai Nairobi
São Paulo Shanghai Taipei Tokyo Toronto

Oxford is a registered trade mark of Oxford University Press
in the UK and in certain other countries

Published in the United States
by Oxford University Press Inc., New York

First published 2003

A catalogue record for this title is available from the British Library

Library of Congress Cataloging in Publication Data
(Data available)

ISBN 0 19 852500 1

10 9 8 7 6 5 4 3 2 1

Typeset by the author in LaTeX

Printed in Great Britain
on acid-free paper by
Biddles Ltd, Guildford & King's Lynn

To Heinz

PREFACE

> The best way to predict the future is to invent it.
>
> Alan Kay

As a reader I always hated prefaces in books, finding them a loss of time and energy and preventing you to reach the interesting material inside the book and always skipped them. So now that I have to write one, I am in a fix, since I do not really know what to put in it.

After all if you have this book in hand and bother to read this, it is certainly because you already have some feeling that one-dimensional systems are quite fascinating, and that you want to know more about them. But just in case you are a reader who is able to browse leisurely through books,[1] let me just tell you briefly what this book is about.

One-dimensional systems of interacting particles have fascinated theorists for more than 50 years now. They are wonderful systems in which interactions play a very special role and whose physics is drastically different from the 'normal' physics of interacting particles, that is, the one known in higher dimensions. From the theoretical point of view, here again they are quite unique. The one-dimensional character makes the problem simple enough so that some rather complete solutions could be obtained using specific methods, and yet complex enough to lead to incredibly rich physics. Crucial theoretical progress were made and new theoretical tools developed in the 1970's which culminated in the 1980's with a new concept of interacting one-dimensional particles, analogous to the Fermi liquid for interacting electrons in three dimensions: the Luttinger liquid. From an experimental point of view, one-dimensional systems were mostly at the beginning a theorist's toy. Experimental realizations started to appear in the 1970's with polymers and organic compounds. But in the last 20 years or so we have seen a real explosion of realization of one-dimensional systems. The progress in material research made it possible to realize bulk materials with one-dimensional structures inside. The most famous ones are the organic superconductors, the ladder compounds, and the spin compounds. At the same time, the tremendous progress in nanotechnology allowed to obtain realizations of isolated one-dimensional systems such as quantum wires, Josephson junction arrays, edge states in quantum hall systems, and nanotubes. Last but not least, the recent progress in Bose condensation in optical traps offer great promises for the future in realizing one-dimensional systems of fermions or bosons with unique properties. These experimental developments have of course triggered a

---

[1] I am afraid that this is an endangered species, but I certainly hope it will never disappear!

corresponding burst of theoretical activity and our understanding of such systems has considerably progressed during this period. New theoretical tools have been developed and new concepts have emerged.

The book is intended to present this fascinating one-dimensional physics. On the theoretical side, some of the tools used to treat one-dimensional systems are quite specific and are thus not part of the standard package of many-body physics that is usually taught. The first part of the book thus gives a step-by-step introduction to the techniques and concepts in the field, while pushing it to the frontier of today's research. Given the amount of material it was of course impossible to be exhaustive. I thus preferred to make a selection of techniques, choosing the ones I knew best and that I used extensively for one-dimensional systems. I apologize in advance to those whose pet subject or technique I could not include in this book. To enjoy the technical sections, and although some brief summary is given in the book, some basic knowledge of many-body physics is suitable (corresponding roughly to the first three chapters of Mahan, 1981). For some of the one-dimensional techniques (e.g. Bethe-ansatz or conformal invariance) only a basic introduction is given in this book. For the readers who want to go further than what is given in this book, many references to review papers or more specialized books have been given in the corresponding chapters. Finally, some quite sophisticated or specialized techniques (e.g. non-abelian bosonization, Ising and WZWN models) are not treated in this book. Given the space constraints, and my quite limited knowledge of these methods, I felt it was impossible for me to make a decent pedagogical introduction to them. I thus chose to skip them completely and to focus on the others, which I could hopefully explain better. For the reader who wants to know more on these I recommend the very nice book by Gogolin, Nersesyan and Tsvelik (Gogolin *et al.*, 1999) where these techniques are explained in detail. The second part of the book is devoted to a study of the various physical realizations of one-dimensional systems. Most of the technicalities have been removed in these sections to directly focus on the physics. Given the multiplicity and importance of experimental realizations of one-dimensional systems I have tried to cover, however briefly, all main realizations.

Although I have tried to make every effort to eradicate minus sign problems and other factors of two in the formulas, I am sure that many mistakes remain in this book. I will thus maintain on my web page[2] a list of errors. This should avoid a few number of persons, including me, many useless sleepless nights.

That's all folks! Now just fasten your seat belts, turn the page and enjoy your trip in the one-dimensional world.

---

[2] Located on the Web site: dpmc.unige.ch

How to read this book

Let me tell briefly what each chapter contains to help the reader find his/her way in this one-dimensional maze.[3] This book can be read 'a la carte'. Depending on your knowledge of theory, and on what you expect to find in this book, the chapters should not be read in the same order.

All (most of) the technology is concentrated in the first five chapters and in the appendices. These parts do not focus on any specific physical system but give the tools needed to tackle any[4] one-dimensional problem. I have tried as much as possible not to assume any previous theoretical knowledge from the reader and to explain in a detailed and hopefully physical way the various theoretical techniques used. These chapters are thus intended to teach the techniques at a slow pace. Here are the contents in greater detail:

- Chapter 1: An introduction to the basic ideas of interacting electrons and the peculiarities of one dimension. The first and second parts of the chapter are simple and should be read since the basic ideas explained here (without any fancy techniques) recur in the rest of the book. The third part explains the fermionic techniques used to tackle one dimension. It is there for the professional[5] of standard many-body physics. If you do not known the book by Mahan (1981) by heart, just *skip this part.* You will not need it for the rest of the book, and it could seriously impair your love of one-dimensional systems. If you *do know* diagrammatic theory then read it since it contains very interesting solutions and will help you make the link with the rest of the book.
- Chapter 2: Explains the very technique of bosonization that will be used repeatedly in the rest of the book. It is a systematic, but step by step, introduction to the technique. If you want to become a professional in the field and use the technique, roll up your sleeves, take a deep breath, and go through the chapter. If not read the next chapter first, which is less systematic but explains the same ideas in a much more physical way. You will then come back to this chapter when you feel ready.
- Chapter 3: This chapter explains using very low level techniques the concept of Luttinger liquid. It is a must to have an idea of the physics of one-dimensional systems. If you do not want to know the technique in detail but simply the physical ideas, this chapter is enough. If you want more you will find it in Chapter 2.
- Chapter 4: This chapter discusses more sophisticated technical and physical situations. It also presents a certain number of pitfalls into which the

---

[3] Which proves that a one-dimensional world can also be a maze.

[4] Well, most of them anyway...

[5] You know, Feynman diagrams and all that.

bosonization beginners[6] can fall. You can read this chapter at your leisure or when you encounter the corresponding situation in one of the physical systems you are interested in.

- Chapter 5: This chapter presents the various microscopic methods used to tackle one-dimensional systems: the famous Bethe-ansatz and numerical techniques. Only for theorists or for long winter nights.

The remaining chapters deal with specific class of systems and discuss some experimental materials. They contain little technique and use the results of the previous technical chapters for the main formulas. You can thus read them directly if you do not care how the results were obtained (I still recommend reading the beginning of Chapter 1 and of Chapter 3 though). These chapters need not be read in sequence since they are independent of each other (with some correlations). You can probe the ones that are relevant to your own interests.

- Chapter 6: All about spin systems. Chains, frustration, spin-Peierls transition, ladders, and quasi-one-dimensional systems, and the corresponding experimental realizations.
- Chapter 7: Single chain fermionic systems. Mostly models: Hubbard, t–J, extended Hubbard. The corresponding experimental systems are discussed in Chapter 8. A discussion on transport in fermionic systems and the Mott transition.
- Chapter 8: Coupled fermionic chains and quasi-one-dimensional fermionic systems. Application to ladders (telephone number compounds) and to organic conductors.
- Chapter 9: Effects of disorder in fermionic systems. Introduction to replicas. Anderson localization in interacting fermionic systems. Application to quantum wires.
- Chapter 10: Boundaries, isolated impurities, and constrictions. Discusses mostly the mesoscopic realizations of Luttinger liquids such as carbon nanotubes and edge states in the quantum hall effect, for which such problems are important.
- Chapter 11: Other examples of Luttinger liquids. A break from Fermi statistics: the life of the one-dimensional bosons. Discussion of Josephson junction arrays and Bose condensates. Applications of bosonization to the study of quantum impurities in three-dimensional Fermi liquids.[7] X-ray, Kondo, and multichannel Kondo problems.

---

[6] Also known as bozos.

[7] I do not know whether this moves up or down in dimension since this is in fact a zero-dimensional problem.

ACKNOWLEDGEMENTS

I would like to express my thanks to all who contributed during the years to my understanding of correlated quantum systems and Luttinger liquids through collaborations or discussions. This includes especially all my colleagues at the Laboratoire de Physique des Solides in Orsay, where most of my own research on one-dimensional systems was carried out in the 20-year span I worked there. The exceptional research environment of 'Building 510' made life quite exciting. Halfway through the writing of this book, I made a move to the DPMC at the University of Geneva: I wish to thank my new colleagues there for their warm welcome and for making the transition as smooth as possible. Their enthusiastic interest for correlated systems in general and low-dimensional systems in particular is a promise that life after dis-location will continue to be as exciting as ever. The material presented in this book has directly benefited from constant interactions with talented students and postdocs, with whom I had the lucky chance to collaborate in the field of one-dimensional systems. I want to thank them especially for these enjoyable interactions and discussions. Needless to say I also benefited immensely from incalculable contributions from many colleagues around the world. Although the list of those I would like to thank is too long to fit here, I will mention in particular N. Andrei, C. Bourbonnais, R. Chitra, L. Degiorgi, A. Georges, D. Jérome, P. Le Doussal, K. Le Hur, A.J. Millis, E. Orignac, R. Shankar, B.S. Shastry, A.M. Tsvelik, and C.M. Varma.

I also would like to thank the brave hearts who had the courage to work through the unfinished drafts of this book, to correct mistakes,[8] and to make invaluable suggestions improving both the shape and the contents of this book: C. Bolech, M.A. Cazalilla, C. Kollath, K. Le Hur, J.S. Meyer, E. Orignac, K. Schönhammer, G. Seelig, D. Senechal, and R. Shankar.

Before writing this book I wondered why authors thanked their family. Now that I think I know why, I would like to thank Marido and Gabriel for being so patient during the completion of this one.

Finally, there is one person who deserves a very special place in these acknowledgements: Heinz Schulz. Heinz first introduced me to the beauty of one dimension, and patiently taught me bosonization during my PhD. In addition to being an excellent advisor and later a very dear friend, he was a remarkable scientist, who made major contributions to the field, spelt in a truly impressive style made of a combination of physical arguments and elegant calculations. I would like to dedicate this book to him.

---

[8] I assume full responsibility for the remaining ones.

CONTENTS

1

PECULIARITIES OF $D = 1$

'Toto, I have a feeling we're not in Kansas anymore'
Dorothy, 'The Wizard of Oz'

Interactions among particles are what endows the one-dimensional world with its special properties. Indeed, there are little differences between, for example, one-dimensional free electrons and their higher-dimensional counterparts (Ziman, 1972). Before we embark on our one-dimensional journey, let me briefly recall the effects of interactions in higher dimensions. This will allow to set the stage, present the basic concepts and ideas of Fermi liquids, and to understand very simply why one-dimensional systems are so different. This will serve as a useful reference when studying their physics.

1.1 Crash course on Fermi liquids

When looking at the order of magnitude of the Coulomb interaction in solids, one notices that for typical systems it is neither dominant nor negligible compared to the kinetic energy. This is what makes interactions so difficult to treat: strictly speaking one cannot really use any perturbation theory. Of course, to have a qualitative idea of the effects of interactions the theorist usually considers models for which the interactions can be seen as adjustable parameters. The effects of interactions in 'high'-dimensional systems[9] has been masterfully explained by Landau's Fermi liquid theory (Landau, 1957*a*; Landau, 1957*b*; Landau, 1958). This theory has been the cornerstone of our understanding of interacting Fermi systems for the last 50 years and has been explained in detail in many textbooks (Nozieres, 1961; Abrikosov *et al.*, 1963; Pines and Nozières, 1966; Mahan, 1981). I will thus give here only a caricature of this theory and refer the reader to the aforementioned textbooks for more details.

The important characteristics of the excitations of a free electron gas are as follows. At $T = 0$ all states up to the Fermi energy are occupied (see Fig. 1.1). So the occupation n_k of a state with momentum k has a *discontinuity* at the Fermi surface. For free electrons the amplitude of the discontinuity is 1. The excitations of the system, compared to the ground state, consist in adding particles with a well-defined momentum k. These excitations have a well-defined momentum k and energy $\epsilon(k)$. They have an infinite lifetime since they are eigenstates of the Hamiltonian. To characterize these excitations one can introduce the spectral function $A(k, \omega)$, which is the probability to find a state with a frequency ω and

---

[9] In this book anything above one will be high dimensional.

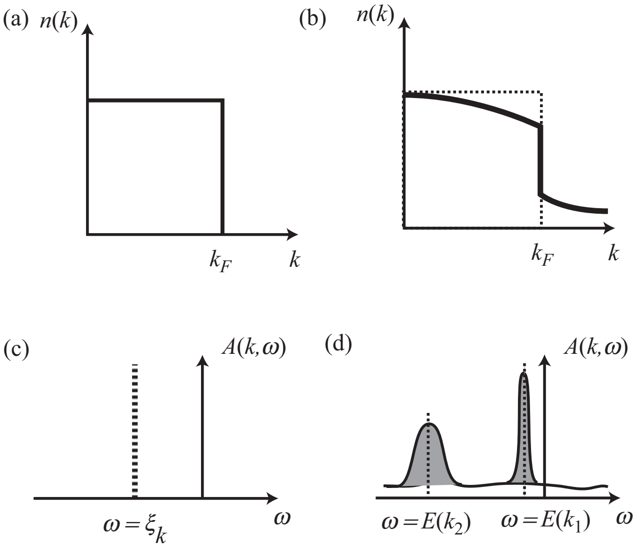


FIG. 1.1. (a) Free electrons: the occupation n_k has a discontinuity of amplitude 1 at the Fermi surface. The spectral function $A(k, \omega)$ (see text) is a delta function peak, showing excitations without damping and a well-defined frequency–momentum relation $\omega = \xi(k)$. The excitations are made of the individual particles of a given momentum. (b) Fermi liquid: it is *essentially* similar to a free electron gas, with some differences. The occupation n_k still has a discontinuity at the Fermi wavevector $k = k_F$, but with a reduced amplitude $Z < 1$. The excitations become sharper when they get closer to the Fermi surface. The total weight in these excitations (quasiparticles) is Z. The quasi-free excitations (quasiparticles) are electrons dressed by the particle–hole excitations of the electron gas.

a momentum k. For free electrons $A(k, \omega) = \delta(\omega - \xi(k))$, where $\xi(k)$ is the energy relative to the chemical potential $\xi(k) = \epsilon(k) - \mu$.

What remains of these nice properties when interactions are switched on in the system? The remarkable result of Fermi liquid theory is that not much changes and that the properties of the system remain *essentially* similar to those of *free* fermionic particles. The elementary particles are not the individual electrons anymore, but electrons dressed by the density fluctuations around them. Just as in electrodynamics an electron dresses by surrounding itself by a photon cloud, here the electrons surround themselves with particle–hole excitations of the ground state (see Fig. 1.1). Since these excitations are made of an electron plus density fluctuations they behave as fermions. These individual objects, called quasiparticles, can be considered as essentially free. Of course, this is only a caricature and the situation in a Fermi liquid is more complicated. Residual interactions exist between the quasiparticles, described by the so-called Landau parameters, and they have an effect in some quantities like susceptibilities. But this caricature will be enough for our purposes. Thus, the occupation number n_k of a state with momentum k still has a *discontinuity* at the Fermi surface. The amplitude of this discontinuity is not 1 anymore but a number Z that represents the 'fraction' of the electron that remains in this quasiparticle state (I will give a more precise definition below). The more interacting the system is, the more scrambled it is and thus the smaller the discontinuity (see Fig. 1.1). The quasiparticles have a 'well-defined' relation between frequency ω and momentum $\omega = E(k)$, which simply reflects the fact that the wavefunction of a quasiparticle has a time dependence $e^{-iE(k)t}$. Of course, $E(k)$ is not the bare energy of an electron. Since one is close to the Fermi surface one can linearize the dispersion in powers of $k - k_F$

$$E(k) \simeq E(k_F) + \frac{k_F}{m^*}(k - k_F) \tag{1.1}$$

which defines the parameter m^*. For free electrons, $m^* = m$ is the mass of the electron. Close to the Fermi surface, changing the dispersion from $\epsilon(k)$ to $E(k)$ simply changes the mass of the excitation from m to a new mass m^*. This is the only change needed. The Fermi momentum is unchanged (for a spherical Fermi surface) thanks to a conservation law known as Luttinger theorem. This theorem states that although the shape of the Fermi surface can be affected by interactions the volume enclosed by the Fermi surface is an invariant (Nozieres, 1961; Abrikosov *et al.*, 1963; Pines and Nozières, 1966; Mahan, 1981). For a system invariant by rotation symmetry (spherical Fermi surface) this imposes that k_F cannot change.

Because they are not completely free the quasiparticle excitations also have a lifetime τ. Thus, the Fourier transform of the time dependence of the wavefunction of the excitation ($t > 0$)

$$e^{-iE(k)t}e^{-t/\tau} \tag{1.2}$$

is not just a delta function but a Lorentzian of width $1/\tau$ centered around $\omega = E(k)$ (see Fig. 1.1). When one looks at energies closer and closer to the Fermi level, $E(k)$, which measure the difference in energy relative to the Fermi level, goes to zero. If the lifetime was constant, the damping time τ would become smaller than the period $1/E(k)$ and thus the excitations would become overdamped. They could not be interpreted as particles similar to free electrons. Fortunately, the lifetime is due to the scattering between quasiparticles. Since close to the Fermi energy quasiparticles have less and less phase space to scatter, the lifetime *diverges* when one goes closer to the Fermi level. Landau remarkably has shown from simple phase space arguments that the lifetime diverges in $d = 3$ as $1/E(k)^2$. Thus, the lifetime is *always* larger than the period close the Fermi level. In fact, the excitations become better and better defined when one approaches the Fermi level. For most properties one can simply ignore the lifetime. It is only important when dealing with properties such as transport. Because it varies as $1/E(k)^2$ it naturally gives a resistivity proportional to T^2.

Having essentially free excitations is what makes Fermi liquid theory work and the concept of quasiparticle so useful. In condensed matter, one is very often for practical purposes ‘close’ to the Fermi level. The Fermi energy is about $10000K$ in normal metals and one works at temperatures of the order of 1–$100K$ which is very small compared to the Fermi energy. At these energies the quasiparticles are the correct well-defined and ‘free’ excitations, whereas the individual electrons themselves are strongly coupled. The spectral function $A(k, \omega)$ thus contains (Lorentzian) peaks centered at $\omega = E(k)$ that becomes sharper and sharper when k goes to k_F. The total weight of these peaks is Z and describes the part of the excitations that is in the quasiparticle state. The rest $1 - Z$ is in a continuous background that has no well-defined structure, but can be safely ignored, close to the Fermi surface compared to the well-defined peaks. Since only the sharp excitations have a reason to give a discontinuity in $n(k)$ at k_F (in a similar way than for free electrons), this explains why the jump in $n(k)$ at $k = k_F$ has an amplitude Z and not unity any more as for free electrons.

One can establish by doing perturbation in the interaction that these properties are indeed correct. But the great strength of Landau’s theory resides in the fact that it is *not restricted to weak coupling.* The existence of quasiparticles, which have fermionic nature and the above properties, is extremely robust and relies only on a phase space argument. It can work even for extremely strong interactions. Self-consistent (Landau, 1958) and renormalization (Shankar, 1994) proofs of Fermi liquid theory have been given. To prevent the theory from applying one thus needs either exceedingly strong interactions or special conditions such that some other instability occurs (see, e.g. Varma *et al.*, 2002).

To complete our brief tour of Fermi liquid let us note two additional facts. First, in addition to these individual quasiparticle excitations, other types of excitations exist in an interacting system. One can define collective excitations that describe the response of the system to a disturbance of the density or the spin density. For example, for the charge, if only short-range interactions are present (as is, e.g. the case in helium) this collective excitation is called zero sound, and represents the way a density wave can propagate (with or without damping) in the electron gas. When long-range (Coulomb) interactions are present, this is the famous plasmon excitation (Abrikosov *et al.*, 1963; Pines and Nozières, 1966). The second point is more formal. One should note that the quasiparticle states are not the exact eigenstates of the interacting problem. Indeed, for an interacting system the number of eigenstates is exponentially large. For example, for a lattice system of fermions with spin the total number of states is 4^N where $N = L^d$ is the number of sites, L the linear dimensions of the system and d the spatial dimension. Since quasiparticles states have an energy $E_k = k_F(k - k_F)/m^*$ the distance between two such states is of the order of $1/L$ where L are the linear dimensions of the system. This is much larger than the exponentially small[10]

---

[10] This is for generic interacting systems. Of course, for the free electron gas, the level separation is $1/L$ and each energy level is highly degenerate. Interaction lifts in general this huge degeneracy between exact eigenstates.

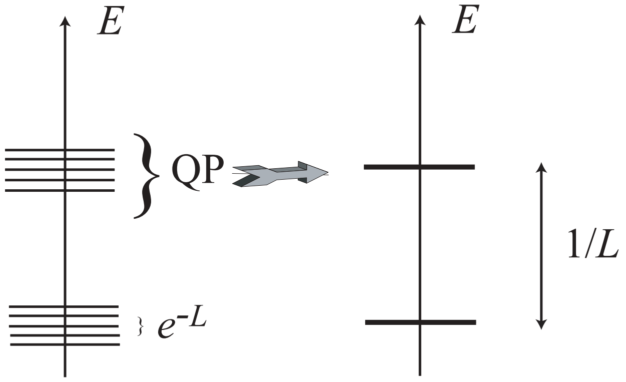


FIG. 1.2. A quasiparticle 'state' is in fact made of a very large number of *exact* eigenstates of the interacting system. The separation in energy of these states is exponentially small in the system size L and thus irrelevant physically, for reasonable systems. The cluster of all these states form the quasiparticle with its average energy and lifetime (inverse of the broadening in energy).

separation in energy between the *exact* eigenstates. As shown in Fig. 1.2, in a Fermi liquid, exact eigenstates thus cluster at given positions in energy. The ensemble of *all* these exact eigenstates form a quasiparticle with its lifetime.

I have purposefully used in this chapter handwaving descriptions. One can formalize the above concepts by introducing the standard many-body description of such systems. Since I will need this formalism later, Appendix A contains a brief summary of the many-body formalism that will be used in this book. I follow the notations of Mahan (1981). In particular, the spectral function is related to the single-particle Green's function by

$$A(k,\omega) = \frac{-1}{\pi} \mathrm{Im}\, G_{\mathrm{ret}}(k,\omega) \tag{1.3}$$

where Im denotes the imaginary part and $G_{\mathrm{ret}}(k,\omega)$ is the retarded Green's function.

1.2 One dimension: Failure of perturbation theory

How much of this beautiful Fermi liquid theory survives in one dimension? Without any calculations it is easy to see that interactions have drastic effects compared to higher dimension. Indeed, in high dimension nearly free quasiparticle excitations exist. In one dimension, as shown in Fig. 1.3, an electron that tries to propagate has to push its neighbors because of electron–electron interactions. No individual motion is possible. Any individual excitation has to become a collective one. This 'collectivization' of excitations is obviously a major difference between the one-dimensional world and higher dimensions. It clearly invalidates any possibility to have a Fermi liquid theory work. This proves that the physical

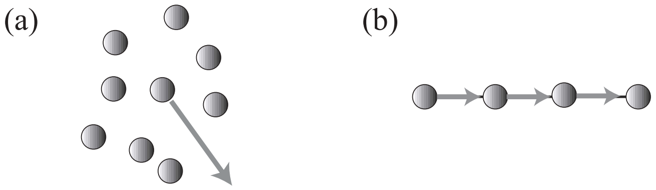


Fig. 1.3. (a) In high dimensions, nearly free quasiparticle excitations, that look nearly as individual particles are possible. (b) In a one-dimensional interacting system, an individual electron cannot move without pushing all the electrons. Thus, only *collective* excitations can exist.

properties of the one-dimensional electron gas are drastically different from the ones of a free electron gas. For fermions with spin this is even worse. Because only collective excitations can exists, it implies that a single fermionic excitation has to split into a collective excitation carrying charge (like a sound wave) and a collective excitation carrying spin (like a spin wave). These excitations have in general different velocities, so the electron has to 'break' into two elementary excitations. These properties, quite different from the ones of a Fermi liquid, will be the essence of the Luttinger liquid, examined in Chapter 3.

Let us be more formal. If some perturbation theory in interaction was attempted, some correlations such as the density–density correlation function would appear in the perturbation theory. Such correlation measures in linear response, the response $\langle \rho(x) \rangle$ of the electron gas to an applied external potential.

$$H_{\text{dens}} = \int d^d x \, V(x,t)\rho(x) \tag{1.4}$$

The susceptibility (see A.33) that measures this response is given by (Ziman, 1972; Mahan, 1981)

$$\chi(q,\omega) = \frac{1}{\Omega} \sum_k \frac{f_F(\xi_k) - f_F(\xi_{k+q})}{\omega + \xi(k) - \xi(k+q) + i\delta} \tag{1.5}$$

where $\delta = 0^+$. Generically, I denote by Ω the volume of the system. f_F is the Fermi factor (see Appendix A for notations). For a system in d dimensions with linear dimensions L, $\Omega = L^d$. Let us focus on the static susceptibility $\chi(q, \omega = 0)$. Normally, it is well behaved at all wavevectors and goes to a constant for $\omega = 0$. One recovers from (1.5) that $\chi(q \to 0, \omega = 0)$ is simply proportional to the density of states at the Fermi level (Ziman, 1972). The standard (positive) compressibility is defined with $V = -\mu$ compared to (1.4), so with our conventions χ is negative. For finite q its behavior is controlled by the way the energy $\xi(k)$ depends on k. When k is on the Fermi surface $\xi(k) = 0$, if one can find a wavevector Q such that both $\xi(k)$ and $\xi(k+Q)$ are zero this leads to singularities. In high dimensions, as shown in Fig 1.4, this occurs only for a very limited set of points. Because of the integration over k in (1.5) the singularity of the denominator is

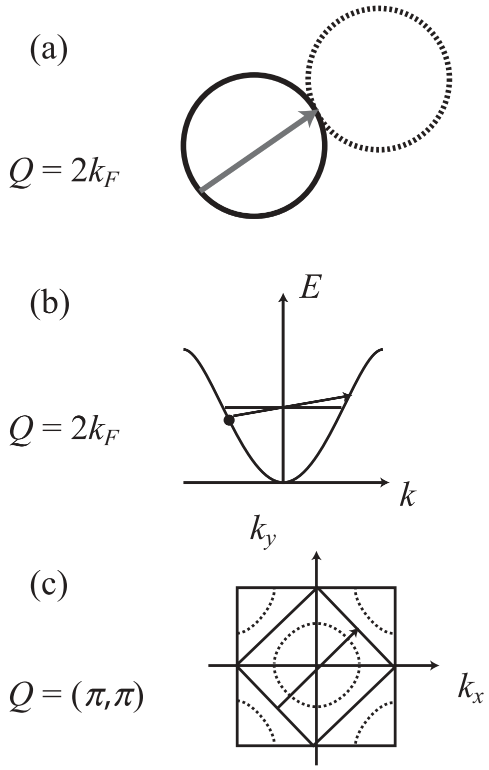


FIG. 1.4. Nesting properties of Fermi surfaces. In high dimensions, a normal Fermi surface leads to nesting only for a very limited set of points and thus to very weak singularities at $Q = 2k_F$ in the derivatives of the susceptibility (a). In one dimension, the Fermi surface consists of only two points and is thus totally nested. This leads to serious singularities at $Q = 2k_F$ (b). Nesting can also occur for special wavevectors Q in higher dimensions, for example, for a square Fermi surface in two dimensions (c).

smoothed out, and only surfaces as a singularity in the slope of $\chi(q)$. This is the root of such phenomena as the Friedel oscillations (Ziman, 1972) and is also known as Kohn anomaly. The only way to have a stronger singularity is for the Fermi surface to satisfy a nesting property, that is, there exists a wavevector Q such that for a finite domain of values of k the energy satisfies

$$\xi(k + Q) = -\xi(k) \tag{1.6}$$

In the domain for which the property (1.6) is satisfied, eqn (1.5) becomes

$$\mathrm{Re}\,\chi(Q, \omega = 0) = -\frac{1}{\Omega}\sum_k \frac{\tanh(\beta\xi(k)/2)}{2\xi(k)} \tag{1.7}$$

where Re denotes the real part and $\beta = 1/(k_B T)$ is the inverse of the temperature (see Appendix A for notations). This becomes, for an infinite volume

$$\mathrm{Re}\,\chi(Q, \omega = 0) = -\int d\xi\; N(\xi) \frac{\tanh(\beta\xi/2)}{2\xi} \tag{1.8}$$

where $N(\xi)$ is the density of states per unit volume. If the density of states is roughly a constant, as is usually the case close to the Fermi level, or if it does not vanish, (1.8) is dominated by a logarithmic singularity. In (1.8), this singularity is regularized by the temperature. A finite frequency leads to a similar regularization. Equation (1.8) behaves as

$$\chi(Q, \omega = 0) \sim -N(\xi = 0)\; \log(E/T) \tag{1.9}$$

where $E = v_F \Lambda$ is some ultraviolet cutoff, of the order of the range in energy over which the nesting property (1.6) holds and v_F the Fermi velocity.

In high dimensions, it is in general impossible to satisfy the nesting property (1.6), except on isolated points (see Fig. 1.4). Thus, χ itself cannot diverge, but its derivatives are in general singular. In one dimension, however, the nesting property is *always* satisfied. Indeed, close to the Fermi points one can linearize the dispersion relation, regardless of its precise nature

$$\begin{aligned} \xi(k) &\simeq v_F(k - k_F), \quad k \sim k_F \\ \xi(k) &\simeq v_F(-k - k_F), \quad k \sim -k_F \end{aligned} \tag{1.10}$$

because of the inversion properties of practically all systems, the Fermi velocity at k_F has to be the same as the one at $-k_F$. It is thus easy to see from (1.10) that

$$\xi(k + 2k_F) = -\xi(k) \tag{1.11}$$

This is shown in Fig. 1.4. Thus, in one dimension nesting is the rule rather than the exception, *regardless* of the precise dispersion relation. The nesting vector is $Q = 2k_F$. Since the susceptibility diverges at $Q = 2k_F$, one can expect any perturbation theory in the interaction to be singular at this wavevector. The fact that a perturbation theory diverges is an indication that the ground state of the interacting system is quite different from the one you started with (that is, the non-interacting one). We thus recover from this more formal argument that the physical properties of interacting electrons in one dimension, however weak the interaction, are drastically different from the free electron ones.

However, simply having a singularity in some susceptibility is usually not a major problem. A well-known example is the singularity that occurs in the particle pairing, leading to the Bardeen–Cooper–Schrieffer (BCS) superconducting instability. This susceptibility measures the response $\langle \psi^\dagger(x)\psi^\dagger(x') \rangle$ of the system to a potential that would like to create or destroy pairs

$$H_{\text{pair}} = \int dx\; V(x,t)[\psi^\dagger(x)\psi^\dagger(x') + \text{h.c.}] \tag{1.12}$$

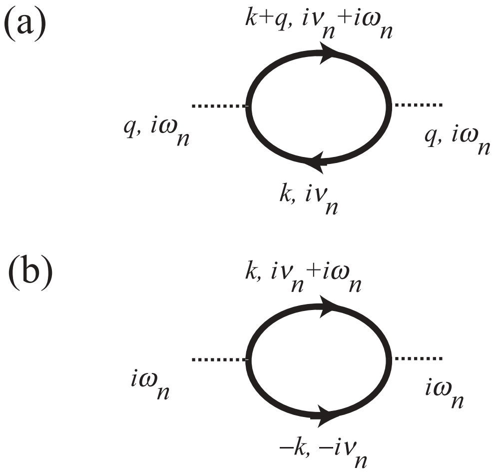


FIG. 1.5. (a) The particle–hole susceptibility (density fluctuations). (b) The particle–particle (pair fluctuations) of the electron gas (only shown for $q = 0$).

If the fermions have spin, one particle has spin up and the other spin down to have a singlet pair and $x' = x$. For spinless fermions, the term (1.12) is killed for $x = x'$ by the Pauli principle. Thus, in that case $x' = x + a$ where a is the smallest distance available on the system (typically the lattice spacing). I restrict here to the case for which the best momentum of the pair is $q = 0$ and show in Fig. 1.5 the corresponding susceptibility at $q = 0$. This is to be compared with (1.4). In that case the potential $V(x,t)$ creates a particle and a hole (the destruction of a particle), instead of creating two particles as in (1.12). In a diagrammatic representation these two susceptibilities are shown in Fig. 1.5. The corresponding susceptibility is given by

$$\begin{aligned}\chi_{\text{pair}}(q=0,\omega) &= \frac{1}{\Omega}\sum_{k}\frac{f(\xi_k)-f(-\xi_{-k})}{\omega-\xi(k)-\xi(-k)+i\delta}\\ \chi_{\text{pair}}(q=0,\omega=0) &\sim N(\xi=0)\ \log(E/T)\end{aligned} \tag{1.13}$$

In high dimension, the response to an external potential is usually treated in a self-consistent way. The susceptibility measures the response to the local potential seen by the electrons, which is the sum of the external one and of the response of the electron gas. If there is an interaction in the system one can treat the interaction in mean field, assuming some kind of order in the system. Typically, one considers order in the density (the so-called particle–hole channel) or order in the pair creation (the so-called particle–particle channel). Any fluctuation imposed by the external potential is thus modified by the presence of the interaction. Let us take an example with a local (Hubbard like, see Section 7.1.1) interaction:

$$H = U \int d^d x \, \rho_\uparrow(x)\rho_\downarrow(x) \tag{1.14}$$

where $\uparrow, \downarrow$ denotes the spin of the electron. If one expects some ordering of the density, the operator $\rho_\sigma(x)$ of the density of each spin species can be rewritten as

$$\rho_\sigma(x) = \langle \rho_\sigma(x) \rangle + (\rho_\sigma(x) - \langle \rho_\sigma(x) \rangle) \tag{1.15}$$

and one can consider that the fluctuations $\delta\rho_\sigma(x) = \rho_\sigma(x) - \langle \rho_\sigma(x) \rangle$ are small. The interaction (1.14) can thus be expanded to linear order in the fluctuations to give

$$\begin{aligned} H &\simeq U \int d^d x [\langle \rho_\uparrow(x) \rangle \langle \rho_\downarrow(x) \rangle + \langle \rho_\uparrow(x) \rangle \delta\rho_\downarrow(x) + \langle \rho_\downarrow(x) \rangle \delta\rho_\uparrow(x)] \\ &= U \int d^d x [\langle \rho_\uparrow(x) \rangle \rho_\downarrow(x) + \langle \rho_\downarrow(x) \rangle \rho_\uparrow(x) - \langle \rho_\uparrow(x) \rangle \langle \rho_\downarrow(x) \rangle] \end{aligned} \tag{1.16}$$

The last term in (1.16) is just a shift of the energy. The two first terms can be seen as an effective potential seen by the electrons of a given spin created by the electrons of the opposite spin via the interaction. For example, for the up electrons the potential is

$$V(x) = U \langle \rho_\downarrow(x) \rangle \tag{1.17}$$

This potential is dependent on the average density at a given point (here of the density of down electrons). In the absence of such interaction the response of the up electrons to an external potential V_{ext} would be

$$\langle \rho_\uparrow(q, \omega) \rangle = \chi^0(q, \omega) V^\uparrow_{\text{ext}}(q, \omega) \tag{1.18}$$

where χ^0 is the susceptibility of the free electron gas (1.5). In the presence of the interaction, one can consider that the linear response (1.18) gives the response to the effective potential. For the up electrons

$$\langle \rho_\uparrow(q, \omega) \rangle = \chi^0(q, \omega) V^\uparrow_{\text{eff}}(q, \omega) \tag{1.19}$$

where the effective potential is the sum of the external one and the one due to the interaction (1.17)

$$V^\uparrow_{\text{eff}}(q, \omega) = V^\uparrow_{\text{ext}}(q, \omega) + U \langle \rho_\downarrow(q, \omega) \rangle \tag{1.20}$$

Of course, the average of the density should be determined from (1.19) (the down electrons obey an identical equation). Putting the two equations together leads to (the argument (q, ω) is understood)

$$\begin{pmatrix} \rho_\uparrow \\ \rho_\downarrow \end{pmatrix} = \begin{pmatrix} 0 & U\chi^0 \\ U\chi^0 & 0 \end{pmatrix} \begin{pmatrix} \rho_\uparrow \\ \rho_\downarrow \end{pmatrix} + \begin{pmatrix} \chi^0 V^\uparrow_{\text{ext}} \\ \chi^0 V^\downarrow_{\text{ext}} \end{pmatrix} \tag{1.21}$$

Let us consider the response to a magnetic field $V^\uparrow_{\text{ext}} = -V^\downarrow_{\text{ext}}$ to probe the magnetic response. Using (1.21) the response to the external potential is

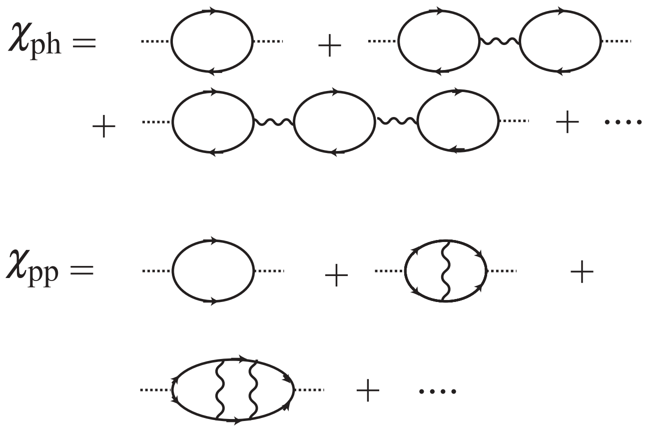


FIG. 1.6. In high dimension, the divergent susceptibilities either in the particle–hole channel (ph) or in the particle–particle one (pp) can be summed up individually, for example, in an RPA approximation, and lead to a finite temperature phase transition. The solid lines are the fermion propagators and the wiggly line is the interaction.

$$\langle \rho_\uparrow(q,\omega)\rangle = \frac{\chi^0(q,\omega)V_{\mathrm{ext}}^{\uparrow}(q,\omega)}{1+U\chi^0(q,\omega)} \tag{1.22}$$

In the presence of interaction, this calculation thus allows to obtain an approximation of the spin susceptibility, which is given by

$$\chi(q,\omega) = \frac{\chi^0(q,\omega)}{1+U\chi^0(q,\omega)} \tag{1.23}$$

This is the well-known RPA (random phase approximation) result that corresponds to summing bubbles or ladders as is shown on Fig. 1.6. A similar calculation can be done for the pairing susceptibility. Summing the diagrams shown in Fig. 1.6 leads to

$$\chi_{\mathrm{pair}}(q,\omega) = \frac{\chi^0_{\mathrm{pair}}(q,\omega)}{1+U\chi^0_{\mathrm{pair}}(q,\omega)} \tag{1.24}$$

If the denominator can be zero there is a finite temperature at which the response diverges. This is usually the signature of a phase transition to an ordered state. For the particle–hole channel, this is the transition to a phase where either the charge or spin density orders. This corresponds either to an antiferromagnetic state (spin order or spin density wave), or to a so-called charge density wave (charge order). The singularity corresponding to a divergence of the particle–particle susceptibility is of course the pairing or BCS superconducting transition. The BCS susceptibility is always divergent, irrespective of the shape of the Fermi surface. This results from the 'nesting' due to time reversal symmetry, that is,

$\epsilon(k) = \epsilon(-k)$. Thus, since $\chi^0_{\rm pair}$ is positive, provided the interaction is negative there is always a transition at a given temperature. In the particle–hole channel in high dimension the susceptibility is not divergent (roughly the density of states), so one needs a finite and strongly repulsive (since χ^0 is negative) interaction to have a transition. This explains why the Fermi liquid state is generically so useful. Since the signs of $\chi^0_{\rm pp}$ and $\chi^0_{\rm ph}$ are opposite one sees that quite generally depending on the sign of the interaction no singularity exists at all in the other channel.

When there is nesting one has to face a special situation, since in that case the particle–hole susceptibility is also always divergent at the nesting wavevector. So there is always for repulsive interactions a transition towards an ordered state in (spin) density. However, as we saw, the particle–particle and particle–hole instabilities in general do not compete, the presence of one usually means that the others are completely killed. We can also expect the RPA to be roughly correct since we know that phase transitions towards ordered states are possible in high dimensions. One dimension is thus quite special. First, nesting always occurs. Second, and more importantly, the RPA result can only be incorrect, since we know that no finite temperature phase transition can exist, even for classical systems (Landau and Lifshitz, 1986). The presence of quantum fluctuations can only make things worse. At a technical level, it means that the simple ladder or bubble resummation of Fig. 1.6 is not enough. There are always two types of divergences, regardless of the shape of the Fermi surface, and *both* should be considered together. Taking into account both bubbles and ladder is known in the diagrammatic lingo as the parquet approximation. Needless to say, it is a nightmarish method, and we will discuss it in more details in Section 1.3. Since these instabilities compete and want to push the system towards different ordered states, they will partly cancel each other, and we can expect some state that will be on the verge of an instability, without ever being able to order completely. One can thus expect the behavior of one-dimensional systems to be close to the critical behavior of systems right at a critical temperature in higher dimension. I will come back to this point in Section 3.4.

Finally, after all these complications, let us note a final peculiarity of one dimension. This time, it will be a great help and in fact at the root of the solution discussed in the next two chapters. A crucial component of the excitations of the electron gas is the so-called particle–hole excitations where an electron is taken from below the Fermi level and promoted above. Since one destroys a particle with momentum k and creates some particle with momentum $k + q$ the momentum of the excitation is well fixed and equal to q. The energy of such an excitation, on the contrary, depends in general on *both* k and q. If one looks at the energy of the particle–hole excitations as a function of their momentum q, one has in general a continuum of energies. In high dimensions, for $q < 2k_F$ one can create particle–hole pairs of arbitrarily low-energy by killing a particle just below the Fermi surface at one point and recreating the particle just above the Fermi surface at another point, as shown in Fig. 1.7. The particle–hole excitations thus

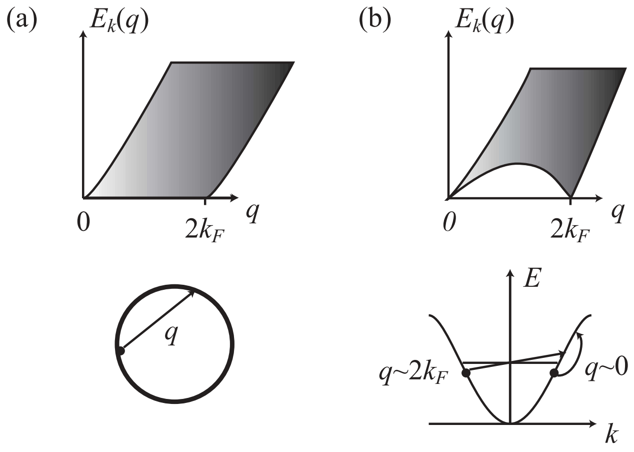


Fig. 1.7. Particle–hole spectrum for two- or three-dimensional systems (a) and for one-dimensional ones (b). In one dimension, contrary to higher dimensions, particle–hole excitations have both a well-defined momentum and energy, for small momentum q.

lead to a continuum extending to zero energy for all q vectors smaller than $2k_F$. In one dimension, the Fermi surface is reduced to two points and one cannot play with angles to increase the momentum q without moving away from the Fermi surface in energy. Since the only way to get a low-energy excitation is to destroy and recreate pairs close to the Fermi 'surface', the only places where the particle–hole energy can reach zero are for $q = 0$ and for $q = 2k_F$. The behavior of the particle–hole spectrum in one dimension is shown in Fig. 1.7. Note the difference with the one in $d \geq 2$. Let us focus on the behavior close to $q = 0$. If we look at excitations at small q the energy of a particle–hole excitation is

$$E_k(q) = \xi(k+q) - \xi(k) \tag{1.25}$$

where $\xi(k)$ should be occupied and $\xi(k+q)$ empty. Let us look at the possible values of $E_k(q)$ for the standard quadratic dispersion

$$\xi(k) = \frac{k^2 - k_F^2}{2m} \tag{1.26}$$

It is easy to check that for $k \in [k_F - q, k_F]$ the average value $E(q)$ of $E_k(q)$ and the dispersion $\delta E(q) = \max(E_k(q)) - \min(E_k(q))$ are

$$\begin{aligned} E(q) &= \frac{k_F q}{m} = v_F q \\ \delta E(q) &= \frac{q^2}{m} = \frac{E(q)^2}{m v_F^2} \end{aligned} \tag{1.27}$$

A similar calculation can be made by expanding the energy around k_F. If one writes

$$\xi(k) = v_F(k - k_F) + \frac{\lambda}{2}(k - k_F)^2 \tag{1.28}$$

Then it is obvious that

$$\begin{aligned} E(q) &= v_F q \\ \delta E(q) &= \lambda q^2 = \frac{\lambda}{v_F^2} E(q)^2 \end{aligned} \tag{1.29}$$

The results (1.27) or (1.29) are noteworthy. They show that regardless of the dispersion relation $\xi(k)$ provided it has a finite slope at the Fermi level: (i) the average energy of a particle–hole excitation is *only* dependent on its momentum q thus the particle–hole excitations are excitations with well-defined momentum q and energy $E(q)$; (ii) the dispersion in energy $\delta E(q)$ goes to zero much faster than the average energy. This is the same situation as the one we discussed for the Fermi liquid quasiparticles. It means that in one dimension the particle–hole excitations are well-defined 'particles' (that is, objects with well-defined momentum and energy), which become longer and longer lived when the energy tends to zero. Because these excitations are made of the destruction and creation of a fermion they are bosonic in nature. These bosonic quasiparticles will just be the key in solving our one-dimensional problem. This remark is at the root of the bosonization method that will be discussed in the next two chapters.

1.3 How to solve

As we have seen in the previous section, one can expect rather peculiar physics in one dimension. This is obvious from the various divergences that plague the perturbation theory in the interaction. These divergences are the way the perturbation theory can tell that the ground state of the interacting system is quite different from free fermions. The first attempt to solve the problem has been made directly in the fermion language. As I hinted in the previous section, this is extremely difficult since it implies the resummation of the various divergences in the perturbation theory. A much more clever method is to use the last remark in the previous section and to reformulate the Hamiltonian in a more convenient basis. This method, known as bosonization, exposed in detail in the next two chapters, will be the cornerstone of this book. However, before we embark on this bosonization solution, I want to expose very briefly in this chapter two of the methods that were put forward to tackle the problem directly in the fermion representation. The goal of this chapter is threefold: (i) A historical interest, since these were the first solutions of the problem and show how to deal with such divergences; (ii) Since these solutions are directly in the fermion representation they nicely complement the other methods. They can and have led to extensions to higher dimensions. (iii) They illustrate some general important concepts such as renormalization.

As you can guess, the rest of this chapter is not for the faint of heart.[11] It requires some knowledge of the many-body theory and diagrammatic techniques. This is the only section of this book with such requirement. Since this section is not necessary to read and understand the rest of the book it can be safely skipped if one wishes to.

1.3.1 *Dzyaloshinskii–Larkin solution*

I will first present a brute force but very elegant solution of the fermionic problem. This is a remarkable calculation (Dzyaloshinskii and Larkin, 1974), and historically a breakthrough in this complicated problem. It also illustrates some of the important properties of the one-dimensional electron gas. For simplicity, this section will deal only with spinless fermions even if the method can be used to treat fermions with spin as well.

If the interactions are much weaker than the bandwidth and if we want to retain the low-energy properties of the system, we want to deal with excitations close to the Fermi surface. In that case we can linearize the dispersion relation close to each Fermi points. We thus replace the original model by one on which the spectrum is purely linear, as shown in the next chapter in Fig. 2.1. This is nothing but the approximation that assumes a constant density of states. Since this approximation is crucial in most methods I will expose it in greater detail in the next chapter, which is supposed to be for a more general readership. I will just give here the main steps. The linearization of the spectrum forces us to introduce two species of fermions: right and left going fermions. The Hamiltonian of the system becomes (Tomonaga–Luttinger model)

$$H = \sum_{k;r=R,L} v_F(\epsilon_r k - k_F) c^\dagger_{r,k} c_{r,k} \tag{1.30}$$

where $\epsilon_R = +1$ for right going particles and $\epsilon_L = -1$ for left going particles. In order to avoid the notation ϵ_r in the following, I use the convention that $r = +1$ for $r = R$ and $r = -1$ for $r = L$ when used in mathematical expressions. What happens far from the Fermi surface is not really important for low-energy processes. Excitations way below the Fermi level are blocked by the Pauli principle and the ones too high in energy cannot be excited anyway. In fact, this is not totally true but I will come back to this point later. We can thus use an arbitrary cutoff procedure to cut the spectrum at large momentum: for example, only allow momentum between $[k_F - \Lambda, k_F + \Lambda]$ (and the same thing on the opposite Fermi point) to exist. Λ in that case can be used to mimic the original bandwidth.

We have the kinetic energy and now need to rewrite the interaction for our new model (1.30). For spinless fermions a typical interaction is

$$H_{\text{int}} = \int dx\, dx'\, V(x - x')\rho(x)\rho(x') \tag{1.31}$$

---

[11] The presentation of these methods fulfils thus another goal. Once you have seen them, you will realize how simple bosonization is!

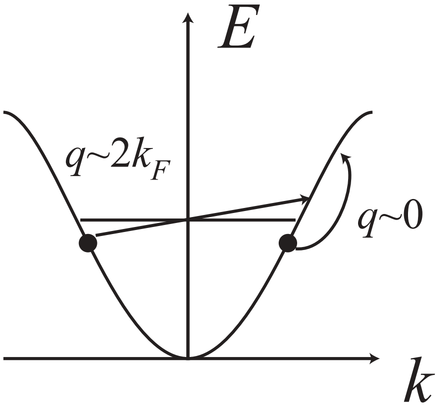


FIG. 1.8. Low-energy processes that the density operator is able to produce. The $q \sim 0$ component of this operator makes particle hole excitations on the same branch, whereas the $q \sim 2k_F$ one transports a particle from one side to the other of the Fermi surface. For the linearized spectrum these two processes correspond, respectively, to the $\psi_R^\dagger(x)\psi_R(x)$ and $\psi_R^\dagger(x)\psi_L(x)$ terms.

The density operator should be written using our left and right going fermions. The single-particle Fermion operator is

$$\begin{aligned}\psi(x) &= \frac{1}{\sqrt{\Omega}}\sum_k e^{ikx}c_k \simeq \frac{1}{\sqrt{\Omega}}\sum_{k\sim -k_F} e^{ikx}c_k + \frac{1}{\sqrt{\Omega}}\sum_{k\sim k_F} e^{ikx}c_k \\ &= \psi_L(x) + \psi_R(x) \end{aligned} \tag{1.32}$$

since only the part of the single-particle operator acting close to the Fermi surface is important for the low-energy properties. The expression (1.32) retains only these important parts. The density operator thus becomes

$$\begin{aligned}\rho(x) &= \psi^\dagger(x)\psi(x) \\ &= \psi_L^\dagger(x)\psi_L(x) + \psi_R^\dagger(x)\psi_R(x) + \psi_L^\dagger(x)\psi_R(x) + \psi_R^\dagger(x)\psi_L(x)\end{aligned} \tag{1.33}$$

The two first and two last terms in the density operators are the expression, using left and right going fermions of the low-energy processes produced by the density operator

$$\rho^\dagger(q) = \sum_k c_{k+q}^\dagger c_k \tag{1.34}$$

since both k and $k+q$ have to be close to one of the Fermi points. These processes are shown in Fig. 1.8.

The interaction (1.31) can be rewritten (see Appendix A)

$$H_{\text{int}} = \frac{1}{2\Omega} \sum_{k,k',q} V(q) c^\dagger_{k+q} c^\dagger_{k'-q} c_{k'} c_k \tag{1.35}$$

One has to keep in mind that the most efficient processes in the interaction are the ones that can act close to the Fermi surface. It means that all momenta in (1.35) have to be close to the Fermi points, that is, close to $-k_F$ or $+k_F$. The fact that in one dimension the Fermi surface is reduced to two points thus allows us to decompose the interaction in three different sectors. These three sectors are shown in Fig. 1.9. The first process g_4 in Fig. 1.9 only couples fermions on the same side of the Fermi surface. The second process g_2 couples fermions from one side of the Fermi surface with fermions on the other side. However, each species stays on the same side of the Fermi surface after the interaction (forward scattering). Finally, the last process g_1 corresponds to a $\sim 2k_F$ scattering (that is, to a backscattering) where fermions exchange sides. Note that for spinless fermions g_2 and g_1 processes are identical since one can exchange the outgoing particles (the particles are indiscernible). This is *not* the case if the fermions have a spin index since in that case one wants the spin to be conserved by the interaction and the processes g_2 and g_1 are different.

Let us now start with our solution. It only works if we do not have g_1 processes. For spinless fermions this is automatically achieved. For fermions with spin this would be a brutal approximation, but there are ways to rationalize this choice. The g_2 process corresponds to a small q matrix element of the interaction, whereas g_1 is clearly the $q \sim 2k_F$ component. For a local interaction the two processes are of the same value, since the Fourier transform of a delta function is a constant. But if the interaction is long-range, or at least non-local, the $q \sim 0$ component is usually much larger. One can thus expect, at least in this case, that keeping only g_2 is a reasonable approximation. The true reason, however, is the fact that the g_1 process is a pain in the neck to treat, as we will see in the next chapter, and one wants to start first with a soluble model. If one retains only the g_2 process then a wonderful simplification occurs. This simplification contains interesting physics that deserves to be discussed it detail. If only g_2 is present then the chirality (that is, the right or left character of the fermions) cannot be changed by an interaction line. Quite remarkably in that case only diagrams with fermion bubbles with *at most* two interaction lines contribute. All the other terms cancel! This is quite extraordinary in the fermion language, and the meaning of this miracle will only become transparent in Section 3.1 (see also Fig. 11.9). To prove this remarkable property, let us examine the two fermionic bubbles with three interaction lines shown in Fig. 1.10. Each time there is bubble (a) in a diagram one has a similar diagram with bubble (b). This would not be the case if there were only two interaction lines. In that case bubbles (a) and (b) would be identical and appear only once in the diagrammatic expansion. Let us evaluate bubble (a). Because the fermion cannot change chirality, the bubble is a product of Green's functions of, say, right going fermions.

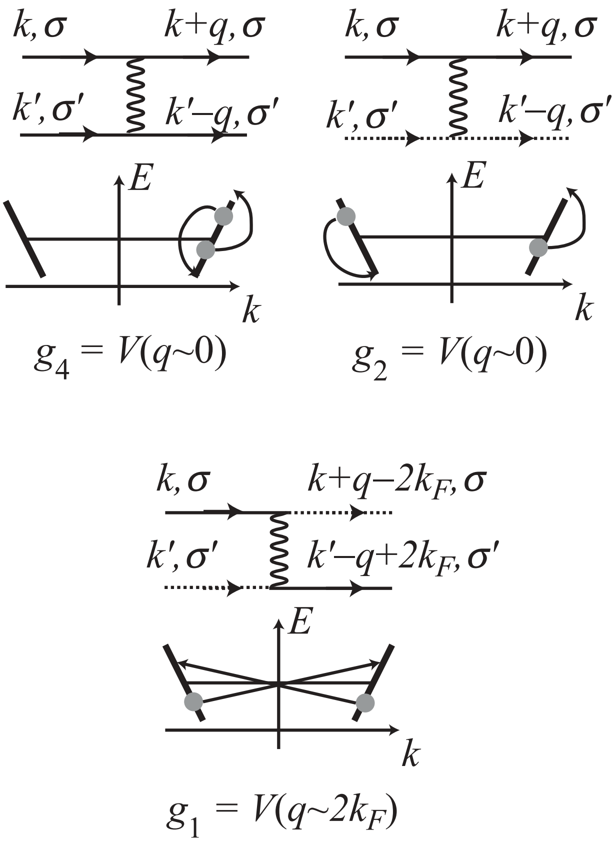


FIG. 1.9. The important low-energy processes of the interaction can be decomposed in three sectors. A full line is for a fermion with a momentum close to $+k_F$ (right going fermion) and a dashed line for a fermion with a momentum close to $-k_F$ (left going fermion). The notation g to designate the different processes is historical (the so-called g-ology). For fermions with spins each interaction can take two values $(g_{\|}, g_{\perp})$ depending on whether the spin σ and σ' of each fermions are equal $(g_{\|})$ or opposite $(g_{\perp})$.

$$(a) \propto \sum_{\nu,k} \frac{1}{i\nu - v_F k} \frac{1}{i(\nu+\omega_a) - v_F(k+k_a)} \frac{1}{i(\nu+\omega_a+\omega_b) - v_F(k+k_a+k_b)} \tag{1.36}$$

Whereas bubble (b) is

$$(b) \propto \sum_{\nu,k} \frac{1}{i\nu - v_F k} \frac{1}{i(\nu+\omega_b) - v_F(k+k_b)} \frac{1}{i(\nu+\omega_a+\omega_b) - v_F(k+k_a+k_b)} \tag{1.37}$$

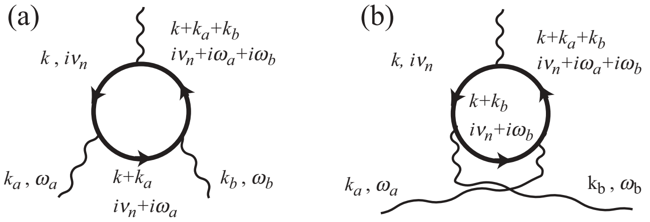


FIG. 1.10. Two fermionic bubbles with more than two interaction lines. These two contributions cancel exactly due to the strictly linear dispersion relation.

Thus, normally (1.36) and (1.37) are quite different. However, because the energy is strictly linear in the momentum and the chirality is conserved, one can rewrite (1.36) as

$$(a) = \frac{1}{i\omega_a - v_F k_a} \sum_{\nu,k} \left[\frac{1}{i\nu - v_F k} - \frac{1}{i(\nu + \omega_a) - v_F(k + k_a)} \right] \times \frac{1}{i(\nu + \omega_a + \omega_b) - v_F(k + k_a + k_b)} \tag{1.38}$$

performing the same operation for bubble (b) one gets

$$(b) = \frac{1}{i\omega_a - v_F k_a} \sum_{\nu,k} \frac{1}{i\nu - v_F k} \left[\frac{1}{i(\nu + \omega_b) - v_F(k + k_b)} - \frac{1}{i(\nu + \omega_a + \omega_b) - v_F(k + k_a + k_b)} \right] \tag{1.39}$$

The second term in (1.39) is obviously cancelled by the first term in (1.38). By shifting $\nu \to \nu + \omega_a$ and $k \to k + k_a$, which one can do unrestrictedly due to the perfectly linear dispersion relation, the remaining term cancels as well. This, of course, assumes that the cutoff Λ is essentially infinite. I will come back to that point in Section 11.2. Thus, bubbles (a) and (b), which are deduced one from the other by permuting two of the legs, cancel perfectly. If there are more than three-legs one can show by induction that the result still holds between the various crossings of the legs of the bubble. Note that when rewriting (1.36) as (1.38) one has one less Green's function depending on ν and k, thus it is similar to a diagram with one external leg less.

As a result, a given diagram can contain bubbles and series of bubbles only. It is as if RPA became exact. This is only true for the *interaction.* For a vertex other terms *do* appear. Let us write in Fig. 1.11 the equations for the effective interactions of the g_4 and g_2 type for our spinless fermions. For fermions with spin the solution is essentially the same (Sólyom, 1979). The equations are

$$\begin{aligned} \Gamma_4 &= g_4 - g_4 \Pi_R \Gamma_4 - g_2 \Pi_L \Gamma_2 \\ \Gamma_2 &= g_2 - g_2 \Pi_R \Gamma_4 - g_4 \Pi_L \Gamma_2 \end{aligned} \tag{1.40}$$

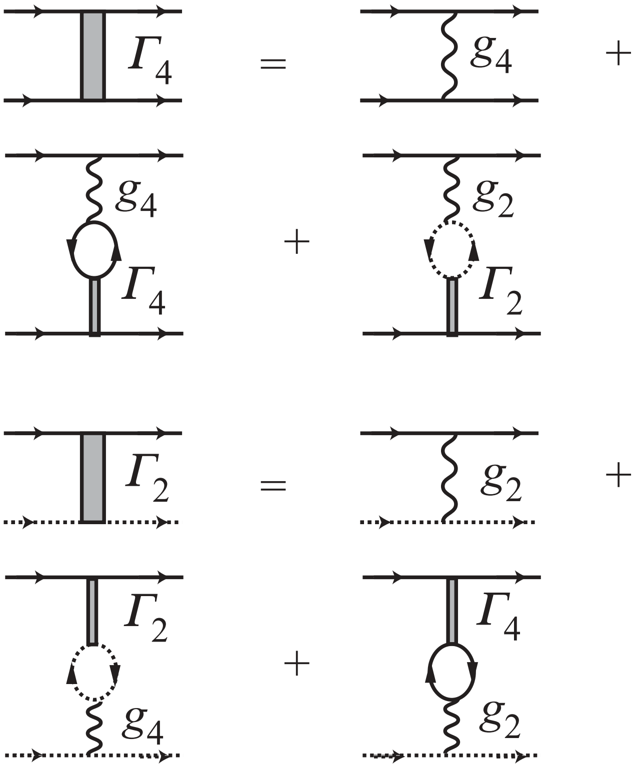


FIG. 1.11. Diagrammatic equations for the effective interactions Γ_4 and Γ_2. Only bubbles and string of bubbles can appear. The full line denotes a right going fermion and a doted line a left going one.

where $\Pi_{R,L}$ are the bubbles made of, respectively, right and left going fermions. For example,

$$\begin{aligned}
\Pi_R(i\nu,k) &= -\frac{1}{\beta\Omega}\sum_{\nu',k'}\frac{1}{i(\nu+\nu')-v_F(k+k')}\frac{1}{i\nu'-v_Fk'} \\
&= -\frac{1}{\beta\Omega}\sum_{\nu',k'}\frac{1}{i\nu-v_Fk}\left[\frac{1}{i\nu'-v_Fk'}-\frac{1}{i(\nu+\nu')-v_F(k+k')}\right] \\
&= -\frac{1}{i\nu-v_Fk}\frac{1}{\Omega}\sum_{k'}[f_F(v_Fk')-f_F(v_F(k+k'))] \\
&= \frac{-k}{2\pi(i\nu-v_Fk)} \qquad (1.41)
\end{aligned}$$

Similarly, one has

$$\Pi_L(i\nu,k) = \frac{k}{2\pi(i\nu+v_Fk)} \qquad (1.42)$$

Equation (1.40) is easily solved giving Γ_4, Γ_2 as a function of the interaction constants g_4, g_2 and the $\Pi(i\nu_n, k)$. Let me give for simplicity the solution for $g_4 = 0$

$$\begin{aligned}\Gamma_2 &= \frac{g_2}{1 - g_2^2 \Pi_R(i\nu_n, k)\Pi_L(i\nu_n, k)} \\ &= \frac{g_2(\nu^2 + v_F^2 k^2)}{(\nu^2 + v_F^2 k^2) - \left(\frac{g_2}{2\pi v_F}\right)^2 (v_F k)^2}\end{aligned} \tag{1.43}$$

Note that there are now poles in the interaction for $i\nu = \pm uk$ where now the velocity is

$$u^2 = v_F^2 \left[1 - \left(\frac{g_2}{2\pi v_F}\right)^2\right] \tag{1.44}$$

This solution has two interesting features. First, contrary to the case of free electrons the interaction vertex has now poles for excitations with a well-defined energy–momentum relationship. Second, the velocity of these poles is not the free Fermi velocity but is renormalized by the interaction (compare with (2.43)). If one had kept the spin degrees of freedom *two* different velocities would have appeared, showing already that our beloved Fermi liquid is on shaky ground. Of course, (1.43) is only the effective interaction. In order to obtain the physics of the problem one needs to relate Γ_2 to some physical quantity. To determine such quantities one would need the exact vertex in addition to the exact interaction. Fortunately, the exact vertex can be extracted using a Ward identity (Dzyaloshinskii and Larkin, 1974; Sólyom, 1979), consequence of the conservation of particles of each chirality.

I stop here since the calculations become really intricate with little gain compared to the simpler methods of the next chapter. This remarkable method, which is purely fermionic, has the advantage to allow some extension to higher dimensions (Metzner *et al.*, 1998).

1.3.2 *Renormalization solution*

The Dzyaloshinskii–Larkin solution is a very nice exact solution of the model. It suffers, however, from many limitations. The most severe one is not to be able to be extended to fermions with spin because of the g_1 processes. I thus present another solution in the fermion representation (Anderson, 1970; Sólyom and Zawadowski, 1974; Sólyom, 1979). This solution is not exact, but its goal is to extract the main singularities from the perturbation theory and sum them. Although not exact such methods have the advantage of being very flexible since they are able to extract the dominant behavior while dropping all irrelevant details. They illustrate the important concept of the renormalization group for fermionic systems relatively simply.

As already pointed out, the perturbation theory in one dimension is plagued by divergences. A very transparent way to see these divergences is to look at

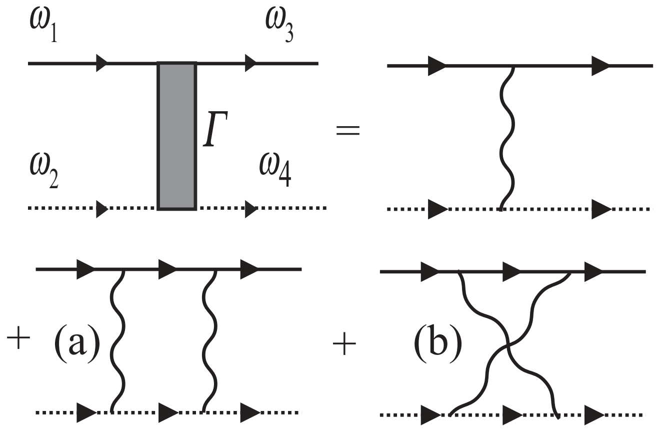


FIG. 1.12. Effective interaction Γ_2 up to second order for spinless fermions. A full line is a right going fermion and a dashed one a left going fermion. The wiggly line is the bare interaction g_2. All external momenta are on the Fermi surface. The second order terms (a) and (b) correspond respectively to the particle–particle and particle–hole channel.

the effective interaction between two particles, identical to the Γ of the previous subsection. Let us first discuss the case of spinless fermions, taking $g_4 = 0$ for simplicity. In that case only the diagrams of Fig. 1.12 exist, to second order. Both diagrams are logarithmically divergent. They can be computed by the standard diagrammatic method and give at $T = 0$ ((a) and (b) refer to the two diagrams in Fig. 1.12)

$$\begin{aligned}(a) &= \frac{1}{2\pi v_F}\log(\omega_{\text{pp}}/E)\\(b) &= \frac{-1}{2\pi v_F}\log(\omega_{\text{ph}}/E)\end{aligned}\tag{1.45}$$

where $E = v_F\Lambda$ is the energy cutoff (the bandwidth). The frequency $\omega_{\text{pp}} = \omega_1 + \omega_2$ for (a), and $\omega_{\text{ph}} = \omega_1 - \omega_4$ for (b). We want the behavior when all these frequencies are small compared to E and of the same order of magnitude. Although one can perform the calculation for any frequency a simple trick is to choose $\omega_1 = 3\omega/2$, $\omega_2 = -\omega/2$, $\omega_3 = \omega_4 = \omega/2$. This satisfies the conservation of frequencies and ensures that $\omega_{\text{pp}} = \omega_{\text{ph}} = \omega$, which simplifies some expressions.

If only type (a) diagrams were present, they could be summed up by summing the ladder type diagrams. If only type (b) diagrams were present they could be summed by summing the bubbles. Since both diagrams are equally divergent they should be treated on an equal footing. One should thus keep in perturbation theory all higher-order terms where one line of interaction in (1.12) is replaced by one of the two diagrams of Fig. 1.12. Such an approximation, which goes beyond the ladder and bubble summation, is known as the parquet approximation (Bychkov *et al.*, 1966). It is a way to sum these logarithmic divergences. Parquet leads to quite formidable integral equations. It is a pre-renormalization group technology that I will not explore further since it has been supplanted by a much more efficient method.

The idea of the renormalization group is quite simple and of course useful in a host of situations. All our troubles come from the fact that we are interested in the *low*-energy properties of the system. The effective interaction is of the form

$$\Gamma = g + g^2 \log(E/\omega) \tag{1.46}$$

where g is some bare coupling constant and ω the energy scale at which we are trying to compute. Because of the log in the perturbation theory, even if we start with a small coupling constant g, if we want the low-energy properties of the system $\log(E/\omega)$ is large and the second term (and a fortiori higher terms) in (1.46) becomes more important than the first term. If, however, we wanted to have *high*-energy properties of the system, that is, work at frequencies (or temperatures etc.) of the order of the cutoff, the $\log(E/\omega)$ would be a simple number of order one, and provided that the coupling constant be small the perturbation theory would be well behaved. This remark has led to the idea of the renormalization group. We start from a theory with some coupling constants g and a cutoff $E = v_F\Lambda$. If we could find another theory with the *same* low-energy properties but with new coupling constants g' and a *lower* cutoff $\Lambda' < \Lambda$, then this new theory would have a less divergent perturbation series (provided, of course, that the new coupling constants g' remain small) for a fixed ω. In doing so we have eliminated some unimportant degrees of freedom and thus have a new theory that is much closer to the one that describes the important degrees of freedom, that is, the low-energy ones. We can thus hope that the new Hamiltonian will be simpler to understand than the original Hamiltonian. This renormalization technique has been one of the most useful concepts in the study of phase transitions and condensed matter in the last 30 years. It is specially useful in condensed matter given the very large range in the typical cutoffs (a bandwidth, typically 1 eV or $12\,000K$) and the energies at which one observes the physical phenomena (typically the temperature, i.e. a few Kelvin). A common misconception is to believe that the renormalization group allows to solve problems. All it does is help to relate one problem to another, while keeping the low-energy physics of the two problems identical. The hope is that one of the related problems will be easily solvable. This can happen for various reasons: (i) One of the problems is exactly solvable. This is an incredible stroke of luck, but not so rare in one dimension. I will discuss examples of this situation in the next chapters. (ii) There is some approximate solution that becomes quite accurate on one of the Hamiltonians. For example, since the cutoff is reduced, if the coupling constants remain small, the perturbation theory is better and better behaved. It is thus possible to use straightforward perturbation theory. But other approximations are possible, and very often more easy to spot since in the renormalized Hamiltonian irrelevant degrees of freedom have been eliminated.

Let us now see how it works practically. For the spinless problem the effective interaction is

$$\Gamma_{\text{eff}} = g_2 + \frac{g_2^2}{2\pi v_F} \log(\omega_{\text{pp}}/E) - \frac{g_2^2}{2\pi v_F} \log(\omega_{\text{ph}}/E) \tag{1.47}$$

The effective interaction obviously controls all the low-energy properties of the system. Thus, we want to keep Γ_{eff} invariant while changing the cutoff. If we change $E \to E'$

$$\log(\omega_{\text{pp}}/E) = \log(\omega_{\text{pp}}/E') + \log(E'/E) \tag{1.48}$$

and thus

$$\Gamma_{\text{eff}} = g_2 + \frac{g_2^2}{2\pi v_F} \log(\omega_{\text{pp}}/E') - \frac{g_2^2}{2\pi v_F} \log(\omega_{\text{ph}}/E') \tag{1.49}$$

since the logarithmic contributions $\log(E'/E)$ cancel. Note that this is obvious if we use the special values of the frequencies that ensures $\omega_{\text{pp}} = \omega_{\text{ph}}$. Of course, as discussed before a different choice of frequencies will not affect the final RG equations. The new coupling constant is simply in that case

$$g_2(E') = g_2(E) \tag{1.50}$$

since all logs vanish. If we define the cutoff as $\Lambda(l) = \Lambda_0 e^{-l}$ (and a similar formula for $E = v_F \Lambda$) we can make the infinitesimal transformation $l \to l + \delta l$. With this variable (1.50) can be rewritten as

$$\frac{dg_2(l)}{dl} = 0 \tag{1.51}$$

We thus see that the forward scattering g_2 is not renormalized but stays the same at any lengthscale. This is consistent with the analysis of the previous section that g_2 drastically changes the properties compared to a free electron gas.

Before we analyze the physical consequences, and because of the simplicity of the method, let us seize the occasion to go one step further and write the interaction process for the system with spins. In that case because the density is

$$\rho(x) = \rho_\uparrow(x) + \rho_\downarrow(x) \tag{1.52}$$

one should rewrite the processes of Fig. 1.9, with a spin index on each fermion line. Thus, each process g can now be different depending on whether the spin on the two fermion lines are parallel $g_\parallel$ or antiparallel $g_\perp$. For interactions that respect spin rotation invariance $g_\parallel = g_\perp$, but one might want to consider more general processes. To illustrate the method I set $g_4 = 0$ for simplicity. We will examine in detail the effects of such processes in the next two chapters. We are thus left with the g_2 and g_1 processes. Since we have spin, we cannot transform the $g_{1\perp}$ process into a g_2 one. However, a $g_{1\parallel}$ is identical by permutation of two

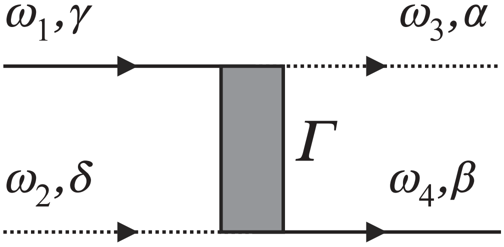


FIG. 1.13. The effective interaction. α, β, γ, δ are the spin indices of the four legs. To ensure that the frequency transfers in the particle–particle and the particle–hole channels are identical, which simplifies expressions, one can take $\omega_1 = 3\omega/2$, $\omega_2 = -\omega/2$, $\omega_3 = \omega_4 = \omega/2$.

fermion lines to a $g_{2\|}$ process. Thus, in defining the effective interaction we have to find a way to separate these two processes. One possible way is to define the effective interaction as

$$\Gamma_{\alpha\beta\delta\gamma} = \Gamma_{1\|}\delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{\alpha\beta} + \Gamma_{1\perp}\delta_{\alpha\gamma}\delta_{\beta\delta}\delta_{\alpha,-\beta} - \Gamma_2\delta_{\alpha\delta}\delta_{\beta\gamma} \tag{1.53}$$

where $\alpha, \beta, \gamma, \delta$ denote the spins on the four fermion lines as shown in Fig. 1.13. This definition fixes all g_2 processes to be isotropic and to put all the difference between $g_{2\|}$ and $g_{2\perp}$ as a $g_{1\|}$ process. It is easy to check that the spinless fermion case is recovered by taking all perpendicular interactions to zero (the two spin species do not talk to each other). This gives $g_2 = 0$ (since g_2 should be isotropic) and $g_{1\perp} = 0$. $g_{1\|}$ plays in that case the role of $-g_2$ in the previous analysis.

With this prescription it is possible to identify each term in the perturbation series. At second order in the interaction the effective interaction is given by the diagrams of Fig. 1.14. Let us write the equation for $g_{1\|}$

$$\Gamma_{1\|} = g_{1\|} - \frac{2g_{1\perp}^2}{2\pi v_F}\log(E/\omega) \tag{1.54}$$

and thus upon a change of the cutoff

$$\Gamma_{1\|} = g_{1\|} - \frac{g_{1\perp}^2}{\pi v_F}\log(E/E') - \frac{g_{1\perp}^2}{\pi v_F}\log(E'/\omega) \tag{1.55}$$

To keep the effective vertex invariant one has thus to change the interaction $g_{1\|}$ as

$$g_{1\|}(E') = g_{1\|}(E) - \frac{g_{1\perp}^2(E)}{\pi v_F}\log(E/E') \tag{1.56}$$

A similar equation for $g_{1\perp}$ gives

$$g_{1\perp}(E') = g_{1\perp}(E) - \frac{g_{1\perp}(E)g_{1\|}(E)}{\pi v_F}\log(E/E') \tag{1.57}$$

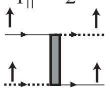
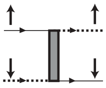
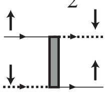
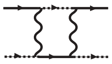
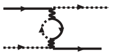
	$\Gamma_{1\parallel}-\Gamma_2$ 	$\Gamma_{1\perp}$ 	$-\Gamma_2$ 
	$-(g_{1\parallel}-g_2)^2$	$-2g_{1\perp}(-g_2)$	$-g_{1\perp}{}^2$
	$(g_{1\parallel}-g_2)^2+g_{1\perp}{}^2$	$2(g_{1\parallel}-g_2)g_{1\perp}$	0

FIG. 1.14. Diagrams entering, up to second order, into the effective interaction between two particles. To obtain the value of the diagram one should multiply the value given in the above table by $A = -\frac{1}{2\pi v_F}\log(E/\omega)$.

Introducing the dimensionless variables $y = g/(\pi v_F)$, the full set of equations for fermions with spins is thus

$$\begin{aligned}\frac{dy_2(l)}{dl} &= -\frac{y_{1\perp}^2(l)}{2}\\ \frac{dy_{1\parallel}(l)}{dl} &= -y_{1\perp}^2(l)\\ \frac{dy_{1\perp}(l)}{dl} &= -y_{1\parallel}(l)y_{1\perp}(l)\end{aligned} \tag{1.58}$$

These equation will be analyzed in great detail in the next chapter. I thus investigate here some simple consequences only.

As is obvious from (1.58), the combination $y_\rho = y_{1\parallel} - 2y_2$ is invariant under renormalization. This combination is the equivalent of (1.51) for spinless fermions. As we will see in the following chapters this is the combination of interactions that controls the charge sector of the system (see (2.105)). The interactions $y_{1\parallel}$ and $y_{1\perp}$ control the spin sector, and have non-trivial renormalization. As expected, for a system with spin rotation invariance $y_\parallel = y_\perp$, the spin rotation invariance is preserved under renormalization. For a system with such spin rotation invariance, the equation for $y_1 = y_{1\parallel} = y_{1\perp}$ is

$$\frac{dy_1(l)}{dl} = -y_1^2(l) \tag{1.59}$$

whose solution is obviously

$$y_1(l) = \frac{y_1(l=0)}{1 + y_1(l=0)l} \tag{1.60}$$

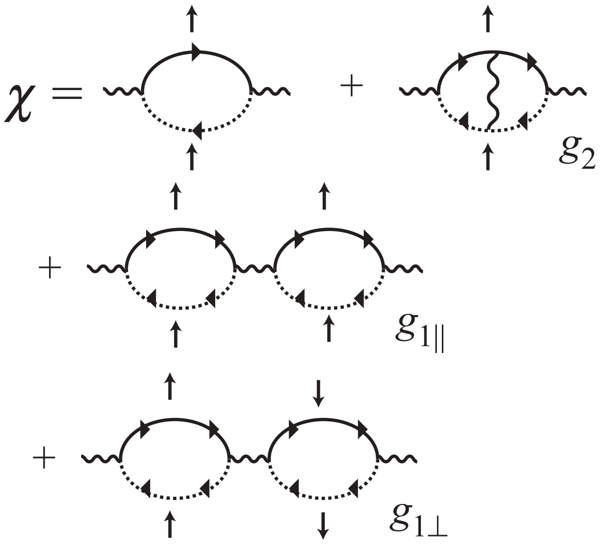


FIG. 1.15. Charge susceptibility, up to first-order in the interaction. One should add to the diagrams shown in the figure the equivalent diagrams where all spins are reversed to get the full χ.

We thus see that if we start from an arbitrary interaction, which gives rise to g_2 and g_1 terms, as we focus on the low-energy properties of the system, y_ρ stays unchanged, whereas the backward scattering renormalizes to zero. The low-energy properties of a physical system will thus be similar to the one of a system with *only* forward scattering. We know how to solve such systems, for example by using the Dzyaloshinskii–Larkin method. This is an example where the RG allows us to simplify the original model and realize that the low-energy properties are in fact equivalent to the ones of a simpler model that we know how to solve.

Let us see how one can extract the physical properties of the system directly from the RG itself. Let us compute, for example, the charge susceptibility $\chi(q = 2k_F, \omega)$, which we argued should be singular for the interacting system. It is given by the diagrams of Fig. 1.15. It gives up to first-order in the interaction

$$\chi(Q = 2k_F, \omega) = \frac{1}{\pi v_F} \log(\omega/E)[1 + \frac{1}{2\pi v_F}(g_{1\|} + g_{1\perp} - g_2) \log(\omega/E)] \qquad (1.61)$$

As we will see in Section 4.4, $\chi(Q = 2k_F, \omega)$ does not obey a simple RG equation. The correct quantity to renormalize is (Sólyom, 1973; Sólyom, 1979)

$$\overline{\chi}(\omega) = \pi v_F \frac{d\chi(Q = 2k_F, \omega)}{d \log(\omega)} \qquad (1.62)$$

Thus, $\overline{\chi}(\omega)$ obeys an equation of the form

$$\overline{\chi}(\omega) = 1 + 2y \log(\omega/E) \tag{1.63}$$

where y stands for the various dimensionless coupling constants $g/(\pi v_F)$. Changing the cutoff from $E = E_0 e^{-l}$ to $E' = E_0 e^{-(l+dl)}$ allows us to write the RG equations for $\log(\overline{\chi})$ at first-order in the coupling constants

$$\frac{d \log \overline{\chi}(l)}{dl} = y_{1\parallel}(l) + y_{1\perp}(l) - y_2(l) \tag{1.64}$$

Let us consider the spin isotropic case. In that case we could just substitute the solution (1.60) in (1.64) and integrate. Let us here brutally simplify the equation by setting $y_{1\parallel} = y_{1\perp} = 0$. A more refined treatment will be seen in Section 4.4. Since y_2 is independent of l the remaining equation can be easily solved

$$\log(\overline{\chi}(l)/\overline{\chi}(l = 0)) = -y_2 l \tag{1.65}$$

We want to obtain $\overline{\chi}(l = 0)$, that is, the correlation function for the original cutoff. We should just find an l such that we can compute $\overline{\chi}(l)$. This is easily achieved when the renormalized cutoff is of the order of ω, that is, for

$$l^* = \log(E_0/\omega) \tag{1.66}$$

in that case $\overline{\chi}(l)$ can straightforwardly be computed from the perturbation formula (1.63) and $\overline{\chi}(l^*) \sim 1$. Thus,

$$\overline{\chi}(\omega) = e^{y_2 l^*} = \left(\frac{E_0}{\omega}\right)^{y_2} \tag{1.67}$$

Integrating one last time, using (1.62) and the fact that for $\omega = E_0$, $\chi(\omega) = 0$ (see (1.61)) one gets

$$\chi(Q = 2k_F, \omega) \simeq \frac{1}{\pi v_F y_2}\left[1 - \left(\frac{E_0}{\omega}\right)^{y_2}\right] \tag{1.68}$$

It is easy to see that an expansion in powers of y_2 reproduces the perturbative result (1.61) as it should. However, we now see that if one looks at the low-energy properties $\omega \to 0$, the correlation function diverges as a *power law*, with *non-universal* exponents depending on the interactions. This is in marked contrast with a Fermi liquid behavior, where $\chi(Q = 2k_F, \omega) \sim \log(\omega/E)$. We will, of course, discuss the physics in detail in the next chapter, after having rederived this result by more convenient methods.

It is possible to push the expansion to higher-order and thus to obtain the exponents in (1.68) in a systematic expansion in the interaction y_2. I refer the reader to (Sólyom, 1979; Chitov and Bourbonnais, 2002) for more on this point.