Hall Response in Interacting Bosonic and Fermionic Ladders

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We use bosonization, retaining band curvature terms, to analyze the Hall response of interacting bosonic and fermionic two-leg ladders threaded by a flux. We derive an explicit expression of the Hall imbalance in a perturbative expansion in the band curvature, retaining fully the interactions. We show that the flux dependence of the Hall imbalance allows to distinguish the two phases (Meissner and vortex) that are present for a bosonic ladder. For small magnetic field we relate the Hall resistance, both for bosonic and fermionic ladders, to the density dependence of the charge stiffness of the system in absence of flux. Our expression unveils a universal interaction-independent behavior in the Galilean invariant case.

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Since its discovery in 1879 [1] the Hall effect has become a remarkable tool for studying solid-state systems. For a single band, the sign of the Hall coefficient R_H, defined as the ratio of the induced electric field to the product of the current density and the applied magnetic field, permits the extraction of the effective charge q and carrier density n, as $R_H \sim -1/nq$ in conventional conductors, indicating whether the carriers are electrons or holes [2]. The Hall effect and its quantum version have thus naturally found applications in metrology for sensitive measurements of magnetic fields and for resistance standards [3]. In noninteracting systems, the Hall effect is interpreted as a manifestation of the topological properties of quantum states, such as the Berry curvature in the anomalous Hall systems [4], and topological invariants in the integer quantum Hall effect [5]. Studies of the dynamical version of the Hall response have also become relevant in fields addressing topological quantum transport [6] and synthetic dimensions [7].

However, the understanding of Hall effect in presence of interactions still remains a fundamental theoretical challenge. With strong magnetic field, the fractional quantum hall effect [8] has revealed excitations with fractional charge and anyonic statistics [9]. In the opposite limit of small magnetic field, a complete interpretation of the Hall effect is still missing and few studies are present [10–12]. Recently a numerical study of the Hall coefficient in a quasi-one dimensional system, has predicted a universal behavior for the Hall coefficient above an interaction threshold [13]. In the case of N-leg ladder systems with $\mathrm{SU}(N)$ symmetry, the Hall imbalance, i.e., the difference of particle number between upper and lower legs, was shown to take the classical value $R_H = 1/n$ [13], a prediction confirmed in a quantum simulation with strongly interacting ultracold fermions [14]. The possibility to reliably measure the Hall effect in strongly correlated ladders thus prompts for an analytical calculation of the Hall polarization and voltage that can shed light on the many-body effect controlling such quantities in a correlated system. Such endeavor has proven, however, elusive for the moment since the most common approximations done to deal with interacting one-dimensional systems lead to an artificial particle-hole symmetry and thus to zero Hall effect.

In this Letter we focus, for a two-leg ladder of bosons or fermions threaded by a magnetic flux, on the study of the Hall effect. We use a bosonization approach [15], but properly taking into account band curvature [10,16]. We obtain analytically both the Hall imbalance and Hall voltage and show that the latter is given by the derivative with respect to density of the logarithm of the charge stiffness of the system without flux. For a Galilean invariant case, this relation leads back to the standard formula [2] related to the carrier density, while in the more general case this remarkable connection between two important transport coefficients encodes the many-body effects triggered by interactions in the Hall response. Beyond the clear potential of our formula to measure Hall voltages and clarify the exotic Hall response of strongly correlated solid-state conductors, our Letter paves the way to the investigation of the topological transport properties of strongly correlated systems of matter.

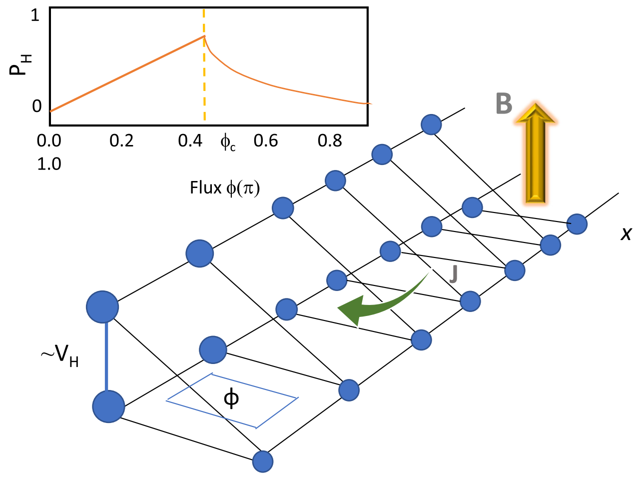

FIG. 1. Two-leg ladder (in real or synthetic dimensions), with an applied magnetic field perpendicular to the plane and current along the x direction. Because of the Hall effect the current is deflected and a Hall voltage can be measured along the rungs. ϕ is the flux of the magnetic field in the plaquette. Although the figure shows also a lattice along the chain direction, the chains can also be uniform. The linear ramp scheme for the measure of the Hall voltage is also shown. The inset shows the qualitative trend of the Hall polarization as a function of the applied flux.

We study the ladder shown in Fig. 1 and populated by bosons or fermions. Such a system can be realized in platforms ranging from cold atoms, both with synthetic dimensions [17] or on a lattice [18–20] to condensed matter such as graphene nanoribbons [21,22] or Josephson junction arrays [23–27].

Let us first examine bosonic systems. Without magnetic field, the generic ladder model has the Hamiltonian:

$$H = H_1 + H_2 - t_\perp \int dx[b_1^\dagger(x)b_2(x) + \mathrm{H.c.}]. \qquad (1)$$

With a lattice along the chains, the integral along x in (1) becomes a discrete sum over the sites j. $H_{1,2}$ describe interacting bosons along the chains, while $t_\perp$ is the tunneling along the rung. Typical realizations of such systems for cold atoms involve contact interactions and are the Lieb-Liniger [28] model for continuous chains [29–31] or the Bose-Hubbard model [32] for chains with a lattice [33–35]. When the ladder is realized using synthetic dimensions [7,36], in addition to the interactions along the chains an interaction across the rung also exists.

Not to be tied to a particular microscopic model, we use the fact that they all have the low energy physics of a Tomonaga-Luttinger liquid (TLL) [15,37–41]. Analyzing the TLL model for the bosonic version of Fig. 1 thus describes all the microscopic models in a unified way. The single chain Hamitonian is ($p = 1$, 2 and we set $\hbar = 1$):

$$H_p = \int \frac{dx}{2\pi}\left[uK(\pi\Pi_p)^2 + \frac{u}{K}(\partial_x\phi_p)^2\right] + \int dx[\alpha(\partial_x\phi_p)^3 + \gamma(\pi\Pi_p)^2\partial_x\phi_p]. \qquad (2)$$

The fields $\partial_x\theta_p = \pi\Pi_p$ and ϕ_p represent, respectively, the collective long wavelength excitations in the pth chain associated to the gradient of the superfluid phase θ_p and to the deviations of the density $\rho_p(x)$ of the bosons $\rho_p(x) - \rho_0 \simeq -(1/\pi)\partial_x\phi_p(x)$ from the average density ρ_0. They obey the commutation relation $[\phi_p(x), \pi\Pi_{p'}(x')] = i\pi\delta_{pp'}\delta(x - x')$. In the first line, the sound velocity u and the dimensionless TLL parameter K can be computed for any given microscopic Hamiltonian [15,41–46]. The parameter K controls the algebraic decay of the correlation functions [15]. The second line corresponds to irrelevant operators that must be retained for the Hall problem. They represent the terms generated by band curvature [47–49] and break particle-hole symmetry. This last point is crucial for a proper description of the Hall effect since particle-hole symmetry immediately implies a zero Hall voltage. The parameters α, γ are given by [50–53]

$$\alpha = -\frac{\partial}{\partial\rho_0}\left(\frac{u}{6\pi^2 K}\right); \qquad \gamma = -\frac{\partial}{\partial\rho_0}\left(\frac{uK}{2\pi^2}\right). \qquad (3)$$

In a Galilean invariant system [37], (3) simplifies to $\gamma = -1/(2\pi m)$, with m the particle mass. Furthermore, in the case of hardcore bosons or spinless fermions with a quadratic dispersion, $\alpha = -1/(6\pi m)$ [47,48]. The generalization to the case with interchain interactions is discussed in the Supplemental Material [54,55].

In addition to the Hamiltonians of the two independent chains (2) the boson tunneling [56,57] between the chains gives

$$H_{\mathrm{tunn}} = -\frac{t_\perp}{\pi a_0}\int dx \cos[\theta_1(x) - \theta_2(x)]. \qquad (4)$$

With a magnetic field and in the Landau gauge, the vector potential along chain p is $A_p = (-1)^{p-1}(Ba/2)$, with a the interchain distance. The vector potential is zero along the rung. As shown in the Supplemental Material [54] this leads to the replacement $\partial_x\theta_p \to \partial_x\theta_p - eA_p$ in (2). The longitudinal current in chain p is obtained from the gauge field dependence of the Hamiltonian $j_p^x(x) = -(\delta H_p/\delta A_p(x))$. We define the Hall polarization [13]:

$$P_H = N_1 - N_2 = \int dx[\rho_1(x) - \rho_2(x)], \qquad (5)$$

which is easily detected [19,58,59]. We introduce the symmetric and antisymmetric basis [56,57] $\phi_{\rho,\sigma} = [(\phi_1 \pm \phi_2)/\sqrt{2}]$; $\Pi_{\rho,\sigma} = [(\Pi_1 \pm \Pi_2)/\sqrt{2}]$ and the corresponding vector potentials $A_{\rho,\sigma} = (1/\sqrt{2})(A_1 \pm A_2)$, yielding the full Hamiltonian $H = \sum_{\nu=\rho,\sigma} H_\nu + H_{\text{hop}} + H_{\text{curv}}$ with

$$H_\nu = \int \frac{dx}{2\pi}\left[u_\nu K_\nu (\pi\Pi_\nu - eA_\nu)^2 + \frac{u_\nu}{K_\nu}(\partial_x \phi_\nu)^2\right], \quad (6)$$

with the tunneling term $H_{\text{hop}} = -\int dx(2t_\perp/a_0)\cos(\sqrt{2}\theta_\sigma)$ and the band curvature term

$$\begin{aligned} H_{\text{curv}} = \int \frac{dx}{\sqrt{2}} &[\gamma\partial_x\phi_\rho(\pi\Pi_\rho - eA_\rho)^2 + \alpha(\partial_x\phi_\rho)^3] \\ &+ \{\partial_x\phi_\rho[\gamma(\pi\Pi_\sigma - eA_\sigma)^2 + 3\alpha(\partial_x\phi_\sigma)^2] \\ &+ 2\gamma(\pi\Pi_\sigma - eA_\sigma)(\pi\Pi_\rho - eA_\rho)\partial_x\phi_\sigma\}, \end{aligned} \quad (7)$$

The parameters are $A_\rho = 0$, $A_\sigma = (Ba/\sqrt{2})$, $u_\nu = u$, $K_\nu = K$. The currents and the Hall polarization become

$$j^x_{\rho,\sigma}(x) = (j^x_1 \pm j^x_2)(x) = -\sqrt{2}\frac{\delta H}{\delta A_{\rho,\sigma}(x)},$$

$$P_H = -\frac{\sqrt{2}}{\pi}\int_{-\infty}^{+\infty}\partial_x\phi_\sigma. \quad (8)$$

Since the Hall effect vanishes with particle-hole symmetry, it is *perturbative* in the band curvature [10,60], in agreement with free fermions calculations. We use this property to build a *perturbative* approach of the Hall imbalance and voltage in the band curvature while retaining the *full nonperturbative* description in the interactions. This allows for the first time for an interacting system to explicitly compute the Hall effect in a regime of weak interchain tunneling. In this framework the equilibrium imbalance is [61]

$$\frac{\langle P_H\rangle}{L} = -\frac{\sqrt{2}\langle T_\tau \partial_x\phi_\sigma(x,0)e^{-\int_0^{+\infty} d\tau H_{\text{curv}}(\tau)}\rangle_{H_\rho+H_\sigma+H_{\text{hop}}}}{\pi\langle T_\tau e^{-\int_0^{+\infty} d\tau H_{\text{curv}}(\tau)}\rangle_{H_\rho+H_\sigma+H_{\text{hop}}}}. \quad (9)$$

We expand the exponentials in (9) to first order in α, γ, leading naturally to an expression of $\langle P_H\rangle$ at first order in the band curvature terms γ, α since the zero order term is canceled by particle-hole symmetry.

Only the last line of (7) contributes. It couples the local density imbalance between the chains $\propto \partial_x\phi_\sigma$ to the difference of current between the chains $\propto (\pi\Pi_\sigma - eA_\sigma)$ and to the total longitudinal current $\propto (\pi\Pi_\rho - eA_\rho)$. Quite remarkably, although the longitudinal total current passing through the system can be a complicated quantity as a function of the gauge potential (flux), the expectation value of the current simply factorizes. Since we ultimately divide by it, it is not necessary to explicitly compute it, leading to further simplifications in our formula since only the antisymmetric correlator is needed. The Hall polarization reads

$$\begin{aligned}\langle P_H\rangle \simeq \frac{\sqrt{2}\gamma\langle j_\rho\rangle L}{eu_\rho K_\rho}\int_0^L dy\int_0^{+\infty} d\tau \\ \times \langle T_\tau[(\pi\Pi_\sigma - eA_\sigma)\partial_x\phi_\sigma]_{(y,\tau)}\partial_x\phi_\sigma(0,0)\rangle_{H_\sigma+H_{\text{hop}}}.\end{aligned} \quad (10)$$

Equation (10) is one of the central results of the Letter: It provides an explicit formula for an analytical or numerical calculation of the Hall imbalance once normalized with the longitudinal current. To finish the calculation we separate the two possible regimes of such a two-leg ladder [57,62–65]: a Meissner phase at low magnetic field in which currents circulate only along the legs ($\langle\Pi_\sigma\rangle = 0$) and, at higher fields, a vortex phase [57], where currents also circulate along the rungs ($\langle\Pi_\sigma\rangle \neq 0$). The two phases are separated by a commensurate-incommensurate transition [66–70].

We first analyze the Meissner regime, for which we can take the limit $B \to 0$. In such phase, the symmetry $\phi_\sigma \to -\phi_\sigma$ and $\Pi_\sigma \to -\Pi_\sigma$, implies $\langle\Pi_\sigma\partial_x\phi_\sigma\partial_x\phi_\rho\rangle = 0$, and the imbalance reduces to (see Supplemental Material [54])

$$\langle P_H\rangle \simeq \frac{\pi^2 Ba\gamma\langle j_\rho\rangle}{2u_\rho K_\rho}\chi_{\sigma\sigma}(q=0,\omega_n=0). \quad (11)$$

Thanks to (7), the first order perturbation in the curvature gives back the expected proportionality both in the total current and in the applied flux. The susceptibility $\chi_{\sigma\sigma}$ measures the charge imbalance in response to a chemical potential difference between the two chains. Within linear response theory [54], $\chi_{\sigma\sigma} = (2K_\sigma/\pi u_\sigma)$, leading to

$$\frac{\langle P_H\rangle}{BL\langle j_\rho\rangle} \simeq \frac{a\pi\gamma K_\sigma}{u_\rho K_\rho u_\sigma}. \quad (12)$$

giving explicitly the Hall imbalance, normalized with the longitudinal current and field, in term of the TLL parameters of the system.

Beyond a certain magnetic field, in the vortex phase, the symmetry becomes $\phi_\sigma \to -\phi_\sigma$ and $\Pi_\sigma \to 2\langle\Pi_\sigma\rangle - \Pi_\sigma$, resulting in $\langle(\Pi_\sigma - \langle\Pi_\sigma\rangle)\partial_x\phi_\sigma\partial_x\phi_\rho\rangle = 0$. As a result

$$\frac{\langle P_H\rangle}{L} \simeq \frac{\sqrt{2}\gamma\langle j_\rho\rangle K_\sigma}{eu_\rho u_\sigma K_\rho}(\pi\langle\Pi_\sigma\rangle - eA_\sigma). \quad (13)$$

The contribution of $\langle\Pi_\sigma\rangle \neq 0$ lowers the Hall imbalance compared with the Meissner phase. As B increases, the expectation value $(\pi\langle\Pi_\sigma\rangle - eA_\sigma) \to 0$ [57], so the Hall polarization tends to zero a signature of the decoupling of the chains at high magnetic field.

We now turn to the calculation of the Hall *voltage*, obtained by adding a potential difference V_H between the two chains [71,72], $-(eV_H/\pi\sqrt{2})\int dx\partial_x\phi_\sigma$, such that $\langle P_H\rangle = 0$. In the Meissner phase, the Hall voltage is

$$V_H = -\frac{\pi^2 \gamma \langle j_\rho \rangle}{e u_\rho K_\rho} Ba = -\langle j_\rho \rangle \frac{Ba}{2e} \frac{1}{u_\rho K_\rho} \frac{\partial}{\partial \rho_0} (u_\rho K_\rho), \tag{14}$$

where we have used the relation (3). Since the longitudinal current density $I_x = \langle j_\rho \rangle / 2$ this reduces to

$$R_H = \frac{a}{e} \frac{\partial}{\partial \rho_0} [\ln(u_\rho K_\rho)]. \tag{15}$$

This formula, which is the second central result of the paper, quite remarkably relates the Hall voltage to the charge stiffness $\mathbb{D}$ [73] (weight of the Drude peak in conductivity, or response of the ground state energy of the systems to a twist in boundary conditions) in the absence of flux. Indeed the charge stiffness is given for a TLL by $u_\rho K_\rho$ [15]. Although we have derived (15) in the absence of interchain interactions we show in the Supplemental Material [54] that this formula remains valid even in their presence.

Several immediate consequences can be extracted from (15). In a Galilean invariant model [37], $uK = (\pi \rho_0 / m)$, and the Hall resistance reduces to

$$R_H = \frac{aB}{e \rho_0}. \tag{16}$$

One thus recovers the "naive" expression of the non-interacting Hall effect (for fermionic particles) for which the Hall effect simply measures the inverse carrier density [2]. This shows analytically for the first time that these results extend to interacting particles and bosonic statistics, provided Galilean invariance holds. Although a general argument leading to (16) in two dimensions was known, it required *both* Lorentz invariance and translational invariance [74]. In the case of an artificial gauge field, the first condition may not be satisfied. More importantly, translational invariance is only satisfied along the chains, preventing the conservation of transverse momentum. In the Supplemental Material [54], we give a simple argument applicable for an artificial gauge fields and hopping in the transverse direction to show that in the Galilean invariant case, $R_H \sim 1/\rho_0$ holds beyond perturbation theory.

Formula (15) thus paves the way to efficiently use the Hall voltage to identify the phase transitions occuring in an interacting system. For example, it allows to distinguish the Meissner phase, where the Hall resistance remains constant and related to the inverse density, and the vortex phase, where the Hall resistance decreases as a function of the magnetic field. Equation (15) also allows to compute, in the absence of Galilean invariance, the impact of the interactions on the Hall voltage. As we show in the Supplemental Material [54] this formula is compatible with numerical calculations of the Hall imbalance in the Meissner phase [72] for the Bose-Hubbard ladder.

In particular (15) shows that the Hall voltage will signal phases in which $\mathbb{D}$ is anomalous such as, e.g., Mott insulating phases. Close a commensurate filling, umklapp operators [15,75] lead to a renormalization of $u_\rho K_\rho$. If the umklapp is irrelevant or away from commensurability we can simply use in (15) the fixed point value $u_\rho^* K_\rho^*$. When the umklapp is relevant, the ladder becomes insulating, $\langle j_\rho \rangle = 0$, and, obviously, $\langle P_H \rangle = 0$. Moving away from commensurability restores the TLL [75,76] via a commensurate-incommensurate transition [66–68,77]. In the vicinity of the transition [68], the renormalized TLL exponent goes to a constant $K_\rho^* = 1/2$, while the velocity vanishes as $u_\rho \sim |\rho_0 - \rho_{0,c}|$. As a result, (15) predicts a *divergent* Hall resistance

$$R_H \sim \frac{a}{e} \frac{\mathrm{sign}(\rho_0 - \rho_{0,c})}{|\rho_0 - \rho_{0,c}|}, \tag{17}$$

with a sign reflecting the nature of the effective carriers. A similar divergence of the Hall resistivity is also obtained in the case of a nearly empty band [78], where $K_\rho \to 1$ while the velocity $u_\rho \sim \rho_0$ and in the case of a nearly filled band of hard core bosons, where $u_\rho \to 1 - \rho_0$. These predictions can be directly tested either in numerical studies or potentially quantum simulations [45]. More properties of the Hall voltage are discussed in the Supplemental Material [54].

Let us finally obtain similar results for the fermionic case, also be realizable in cold atoms [14,79–82] or condensed matter [21,22] experiments. Without interchain hopping, a spin-1/2 fermions chain with repulsive contact interaction is described in the continuum by the Gaudin-Yang model [83,84] and on a lattice by the Hubbard model [85]. Because of the SU(2) symmetric interactions, the Zeeman field leaves the spin modes [86] gapless in contrast with the generic spinless fermion ladder [87,88]. The Hamiltonian for fermions with a quadratic dispersion along the chains reads

$$\begin{aligned} H = &\sum_{p=1,2} \int dx \frac{\psi_p^\dagger(x)(\frac{\hbar}{i}\nabla - eA_p)^2 \psi_p(x)}{2m} - \mu \psi_p^\dagger(x)\psi_p(x) \\ &- t_\perp \int dx (\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1) + U \int dx \rho_1(x)\rho_2(x) \\ &+ \frac{1}{2} \int dx V(x - x')[\rho_1(x)\rho_1(x') + \rho_2(x)\rho_2(x')], \end{aligned} \tag{18}$$

where ψ_p annihilates a fermion in the chain p, $\rho_p = \psi_p^\dagger \psi_p$, A_p is the vector potential in the Landau gauge along the chain p, μ the chemical potential, m the mass, e the fermion charge, $V(x)$ the short ranged-intrachain interaction, U the interchain interaction and $t_\perp$ the interchain hopping. Linearizing the spectrum around the Fermi points, with $\psi_p(x) = e^{ik_F x} \psi_{Rp}(x) + e^{-ik_F x} \psi_{Lp}(x)$, with $k_F = \pi \rho_0$ the Fermi wave vector and $v_F = k_F / m$ the Fermi velocity, we bosonize using standard formulas [87,89,90] (see Supplemental Material [54]) in terms of symmetric $\phi_\rho;\theta_\rho$ and antisymmetric $\phi_\sigma;\theta_\sigma$ fields. The Hall polarization has the bosonized expression

$$P_H = 2\int \frac{dx}{\pi a_0} \cos\sqrt{2}\theta_\sigma \cos\sqrt{2}\phi_\sigma. \tag{19}$$

Without band curvature and magnetic flux, the ground state of the model [87,89] is well known. The field θ_σ is ordered with an expectation value $\langle\theta_\sigma\rangle = \pi/\sqrt{8}$, while the field θ_ρ remains gapless. The curvature terms yield a contribution to the Hamiltonian

$$-\frac{eA_\sigma}{\pi m a_0}\int dx(\sqrt{2}\pi\Pi_\rho - eA_\rho)\cos\sqrt{2}\theta_\sigma \cos\sqrt{2}\phi_\sigma, \tag{20}$$

which results in an expression of $\langle P_H\rangle$ proportional to $B\langle j_\rho\rangle\chi_\perp$, with $\chi_\perp$ the in-plane magnetic susceptibility of an easy axis antiferromagnetic XXZ spin-1/2 chain at wave vector $t_\perp/v_F$. Since the XXZ chain is close to the isotropic point, for large enough $t_\perp/v_F$, the in-plane susceptibility is the same as in the isotropic chain. Up to logarithmic corrections [86,91], $\chi_\perp \sim 1/v_F$, and the perturbative behavior of the imbalance in the fermionic ladder is the same as in the bosonic ladder.

To find the Hall resistance, we follow the same path than for bosons. In the bonding–antibonding basis, after bosonization, the potential difference reads

$$V_\perp = -\frac{eV_H}{\pi a_0}\int dx \cos\sqrt{2}\theta_\sigma \cos\sqrt{2}\phi_\sigma, \tag{21}$$

and we have to determine V_H such that $\langle P_H\rangle_{H+V_\perp} = 0$. To lowest order in $1/m$, we obtain the same expression for R_H as in the bosonic case, (15). We have explicitly checked in the Supplemental Material [54] that at low flux, a relation similar to (15) is valid for a two-leg ladder of noninteracting fermions.

In conclusion, we have derived an analytic expression for the Hall imbalance and Hall voltage for a two-leg ladder threaded by a flux using a bosonization approach retaining band curvature terms. One of our central results relates the Hall resistance to the logarithmic derivative of the charge stiffness with respect to the density. This formula, applicable both for bosons and fermions—as long as there is only one gapless mode—allows a direct and practical calculation of the interaction effects on the Hall voltage. In particular it shows that the Hall resistance diverges close to a Mott insulating phase, on the other side, in the Galilean invariant case, it depends only on density as in the noninteracting case.

The Hall response studied here can be detected in the controllable quantum simulator of a two-leg ladder with interacting bosonic [18–20] and fermionic [92,93] atoms. A first measure of Hall response in a quantum simulator with strongly interacting ultracold fermions has recently appeared [14], opening the way towards further experimental study in different regimes of the interaction and magnetic field. For bosons instead a systematic measurement of the Hall coefficient was shown recently in Ref. [94] and our Letter is compatible with the experimental platforms of Refs. [14,94,95]. Another experimental platform could be Josephson junctions networks [23,25–27,96], where the Hall resistivity could be directly measured. A last possible platform, transmon qubits [97,98] have recently been shown to realize the hardcore boson ladder, and there are recent proposals [99] to realize artificial gauge fields in a transmon ladder. A theoretical challenge is now to extend these results to a larger number of legs for the Hall effect, or so investigate if similar universal relations would also exist for other transport quantities such as the thermopower or Nernst effect [100].

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